A GENERALIZED INPUT-OUTPUT MODEL:

COMBINING DEMAND- AND SUPPLY-SIDE SYSTEMS

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ABSTRACT

The supply-side input-output model has been proposed as an alternative to the demand-side input-output model, especially for the cases of supply shortages in specific commodities. Although several studies have shown the supply-side model to be as valid as the demand-side model on empirical grounds, the former model has been criticized for violating the principles of production theory.

We use the inherent symmetry between the two models to combine them into a single iterative "generalized" input-output model. After deriving the reduced form of the model, we investigate its conditions of consistency and stability. We differentiate two versions of the model where the "sectoral demand-supply mix parameters" operate on the categories of "commodity" and "industry." A procedure for estimating the mix parameters is applied to the U.S. input-output data. We demonstrate that the forecasting performance of the new model compares favorably to both the demand- and supply-side input-output models.

We also extend the model to incorporate flexible prices. Then, we show that the value added component can be endogenized according to a given production function. The model as such corresponds to quantity rationing of intermediate demand with no supply constraints on value added. Finally, we use the relation between the technology of the generalized input-output model and unbalanced growth to construct a dynamic von Neumann-Leontief model of unevenly expanding economy.

The generalized input-output model captures the feedbacks between demand and supply conditions and enables the analyst to control the relative strengths of backward and forward linkages in forecasting. Its applications include the investigation of the relative historical importance of demand and supply forces in economic development, the construction of total linkage indices for the identification of key sectors, and the analysis of the economic impacts of changes in final demand under supply constraints. Two important extensions, not undertaken in this study, are to computable general equilibrium models and multiregional input-output models.

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CHAPTER 1

INTRODUCTION

Quesnay, in his Tableau Economique, the precursor to Leontief's input-output model, likens the economic system to the circular flow of blood in the human body. The economic agents are buyers and sellers at the same time, and sales and purchases are fundamentally linked: they are simultaneously cause and effect. The developments in the field of input-output theory, since it was presented by Leontief over fifty years ago, have principally emphasized only one direction of this flow: that which relates final goods to primary factors. In this demand-side model, the production and value added coefficients are assumed fixed for each sector and a given bill of final demand determines output through backward linkage multipliers. The reverse direction of the flow, that is, the forward linkage relating factors to final goods has generally been neglected by economic theorists. The first study to investigate the nature of this reverse flow within an input-output framework was Ghosh (1958) whose supply-side model is a mirror image of Leontief's original demand-side model. In this framework, allocation (i.e., the distribution of product sales across sectors) is fixed and causality is reversed: a given level of value added determines output through forward linkage multipliers. The conventional bias is turned 180 degrees in the opposite direction.

In the demand-side model, sectors are free to choose the level of intermediate inputs that minimize their cost function, and relative prices adjust to bring the economy to its Walrasian equilibrium. In the supply-side model, on the other hand, institutional rigidities give rise to quantity constraints that affect the level of intermediate inputs for a given sector. The latter is an economy where quantity signals bring about a non-Walrasian equilibrium and is close in spirit to the notion of market disequilibrium.

The objective of this dissertation is to unite these two poles of interindustry analysis in a consistent and stable "generalized input-output model" which is conceived as a convex combination of the demand- and supply-side models. The underlying claim is that the truth lies somewhere in between the two statements: "demand determines output" and "supply determines output." By generalizing the input-output framework over the continuum of its two end points, we hope to capture the whole space of possible combinations of price and quantity signals for each sector. We are principally interested in the allocation of the intermediate inputs and the consequent interindustry equilibrium. Hence, throughout the study, we assume a composite value added category whose "valuation" serves as the numeraire. This simplifies our exposition although it can easily be relaxed. Furthermore, the value added component of production is left outside the realm of quantity constraints. (We can close the model with respect to households, for example, to include labor in our framework.) Value added, in all versions of the model, is either given exogenously or determined endogenously according to a given rule. On the one hand, the practical implication of such a model is improved forecasting and planning based on input-output models (and their extensions, such as computable general equilibrium models); on the other hand, it is also a theoretical and mathematical exercise in building an interindustry general (dis)equilibrium model where the demand/supply bias can be controlled.

We devote a large part of the study to the mathematical formulation of the model, which can also be applied to linear economic systems other than the input-output model.

In Chapter 1, we review the scope and the background of the study, and discuss the "conventional" input-output models. In Chapter 2, we introduce the nominal iterative generalized input-output model, derive its reduced form, and identify its consistency and stability conditions.[1] We distinguish between two linear ways of combining the demand- and supply-side models.[2] We, then, estimate the sectoral demand-supply mix parameters for both versions of the generalized model for the U.S. input-output data and use these results to evaluate the comparative performance of the four different kinds of input-output models in ex post forecasts. In Chapter 3, we discuss input-output models with flexible prices, formulate the flexible price generalized input-output model and briefly discuss two solution algorithms. Chapter 4 is composed of various extensions of the model. First, we show how the model can be made compatible with various production functions. We, then, analyze the implicit relation between technology and unbalanced growth in the long run. Finally, we formulate a simple von Neumann type dynamic growth model based on the technological assumptions of the generalized input-output

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[1] We later show that, the nominal generalized input-output model is a special case of the physical generalized input-output model. We introduce it explicitly, however, for two reasons. First, in practice, the input-output tables are usually available in nominal values evaluated at current prices and are used under fixed value share assumptions. Second, it permits us to present the complexities involved in stages.

[2] We emphasize the "horizontal" version throughout the study because it fits our theoretical structure of quantity rationing, while we carry the corresponding formulations for the "vertical" version parenthetically.

model. We use the generalized von Neumann-Leontief model to compute the possible unbalanced growth paths of the economy along with the price and output proportions that support each path. This is a first step in extending our framework to the dynamic input-output model.

BACKGROUND

The fundamental assumption of the demand-side input-output model is fixed production coefficients. Ghosh (1958) points out that scarcity of factors and/or intermediate goods and lack of unused capacity in some sectors partly invalidate this assumption. Under an environment of shortages and rationing, not all sectors will be able to increase production through proportional increases in all inputs, and some will have to engage in considerable substitution. Furthermore, in an economy characterized by monopolistic/oligopolistic control or centralized planning, that is, an economy with less than competitive markets, firms follow certain allocation principles that are not derived from efficiency in production but from the obligation or incentive to deliver more or less fixed quotas of their output to other firms. In the case of a centralized economy, such a principle might be the maximisation of a welfare function, while in a capitalist economy it might be the long run strategy of retaining markets. An oligopolistic sector is characterized by the stability of relative market shares of the firms over time. Unlike the competitive firm, the objective of an oligopolistic firm is maximizing its sales rather than its short run profits. Especially in smaller countries, it is likely that prices will change to stabilize the allocation coefficients of the sectors characterized by shortages and/or oligopoly. For such circumstances, where supply considerations outweigh demand considerations, Ghosh proposes a supply-side model where it is the allocation coefficients and final demand/output ratios that are assumed to be fixed and the causality runs from factors of production towards final demand. Demand is completely elastic: final consumption and investment respond perfectly to any change in the supply of factors of production. Furthermore, there are no explicit constraints on the possible combinations of inputs in producing a given sector's output.

Giarratani (1980) criticizes the lack of microeconomic foundations of the supply-side model. He points out that although the demand-side model is firmly based on the conventional theory of production, and specifically on Samuelson's substitution theorem, "there is no theory of sales or markets from which we can derive the fixed output hypothesis." Oosterhaven (1981, p.140), similarly argues that the supply-side model is implausible because "input ratios are entirely endogenous and may assume any value depending on the availability of supply," and "value added is not influenced by endogenous production rises." We take a different point of view later by showing the disequilibrium foundations of the (horizontal) generalized input-output model and by endogenizing value added according to a production function that allows substitution between intermediate inputs and value added. This is only possible under flexible prices.

Chen and Rose (1985) show that, through the "similarity" relation between the production and allocation coefficient matrices, the stability of one matrix implies the stability of the other[1] and conclude that the

[1] Two square matrices, A and B, are similar and share the same eigenvalues if they are connected by the relation $A = HBH^{-1}$. We show, in Chapter 4,

supply-side model "can be utilized with the confidence that it will not violate the basic production conditions of its conventional counterpart." Deman (1988, p.815), elaborating on this similarity relation, asserts that "if input coefficients of a Leontief demand-driven model follow a biproportional change, then it will also ensure both stability of supply coefficients in a supply-driven model and consistency between the supply-driven and demand-driven models." Although Oosterhaven (1988, p.206) agrees that "only under conditions of very uneven sectoral growth can one expect a significant difference in stability between two sets (of input and output coefficients)," he argues that the usefulness of the supply-side model must be evaluated by theoretical reasoning and concludes that the supply-side model "eliminates the notion of the production function."

Cronin (1984) distinguishes between two components of an input-output model: the analytical assumption (production or allocation function) and the causal ordering (driven by final demand or value added). He suggests two additional hybrid models: a demand-driven model with fixed allocation coefficients and a supply-driven model with fixed production coefficients. For the former model, output is determined by sectoral output/final demand ratios and for the latter, output is determined by sectoral output/value added ratios. The off-diagonal elements of the inverse matrices of both models are zero and therefore they are not truly interindustry models: no

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that the implicit change in the matrix of production coefficients, A, in a forecast based on the supply-side model is GAG^{-1}, where G is a diagonal matrix of sectoral growth factors. Hence, the matrix of production coefficients change in an inherently stable manner. For a mixed demand-supply model, the forecasted production coefficients matrix lies between A and GAG^{-1}.

linkages are captured in the determination of output. (Oosterhaven (1988, p.208) calls these "degenerate" models). However, the rather trivial hybrid models have important consequences for the choice of causality in the generalized input-output model. Specifically, we can formulate a mixed demand-supply model where the coefficients matrix is a convex combination of the production and allocation matrices, while it can be driven by final demand, value added, or their convex combination.

On the empirical side, several analysts (Augustinovics (1970), Giarratani (1981), Bon (1986)) have shown that neither patterns of variation for the allocation and production coefficients, nor ex post forecasting performance suggest that one model is superior to the other. Helmstadter and Richtering (1982), using the 1960-1975 input-output data for the Federal Republic of Germany, find that the supply-side model significantly outperforms the demand-side model. Furthermore, the stability of the output coefficients over the input coefficients is more pronounced over the long run. They speculate that the fixed output coefficients assumption "compensates partly the columnwise substitution of intermediate products which changes input and output structures as well." Bon (1986, p.235), on the other hand, suggests that "further research should concentrate on combining the two models for the purpose of direct measurement of the relative importance of demand and supply forces," and he warns that "care must be taken to avoid both theoretical and mathematical inconsistency that would result from confounding the two underlying economic principles."

The paradigms of shortage (i.e., excess demand) and oligopoly constitute the two behavioral basis of the supply-side model.[1] However, instances of shortage and oligopoly are usually confined to several sectors and not to the whole economy. Consequently, the use of the supply-side model is not totally justified.[2] The generalized input-output model allows us to assume different analytical assumptions for different sectors. Furthermore, it permits us to combine the two sets of assumptions on a continuous scale for a given sector.

Kornai (1979) differentiates between the demand-constrained and resource-constrained systems, corresponding to capitalist and centrally planned economies respectively. The former is associated with unemployment, while the latter is associated with chronic shortage where bottlenecks of production delimit output. Kornai presents a hydraulic analogy to the latter system where firms with soft budget constraints and irresistible expansion drives "pump out" the slack of the system. In this framework, a budget constraint is soft if the firm is entitled to credit under an "unorthodox and unconservative" financial environment. (He argues that there are signs of softening budget constraints in modern capitalist economy as well and that the degree of softness and hardness should be measured on a continuous scale.) Under this system of "suction," the demand of firms for intermediate inputs and primary factors

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[1] Shortage and oligopoly are associated with the categories "commodity" and "industry" respectively. In Chapter 2, we present two versions of the generalized input-output model, roughly corresponding to this delineation.

[2] See, for example, the empirical studies by Giarratani (1976), Davis and Salkin (1984), and Chen and Rose (1986), which apply the supply-side input-output model to investigate the implications of shortage in one single commodity.

is almost insatiable and a situation is created where "shortage breeds shortage." Firms, as sellers, are confronted by excess demand, and in a bid to meet this demand set unreasonable production goals, in turn, creating further excess demand for inputs. Furthermore, hoarding is the rational response to chronic shortage, further generating shortage. Kornai's resource-constrained system, where sectors are rationed for intermediate inputs, serves as a behavioral basis for the supply-side model under the presence of shortage. At the conclusion of his article, Kornai poses if it is possible "to develop a kind of in between situation, i.e., a convex combination of the two different institutional set-ups and, together with it, such a situation in which there would be neither labor shortage nor unemployment." The generalized input-output model, from the point of view of mixing planning and markets, is the analytical formulation to this normative question.

CONVENTIONAL INPUT-OUTPUT MODELS

We first present the standard input-output models in Equations (3) and (4). Then, in order to present the generalized model in a comparative framework to the existing demand- and supply-side models, we give a slightly different than usual presentation of the latter models in Equations (5) and (6). We express the solutions to all input-output models in terms of projected coefficient matrices along with a constraint that describes the relationship between the base year and projected coefficient matrices over the construction of the interindustry flow matrix. This presentation will facilitate our comparisons of the conventional and the generalized input-output models. We start with an input-output table, expressed in some kind of homogeneous units. The following identities describe the system, before any analytical and causal assumptions are made:

(1) $$Zi + Y = X,$$

(2) $$Z^{T}i + W = X,$$

where

Z = $n \times n$ matrix of interindustry flows, z_{ij},

Y = $n \times 1$ column vector of sectoral final demands, y_i,

X = $n \times 1$ column vector of sectoral outputs, x_i,

W = $n \times 1$ column vector of sectoral value added, w_i,

i = $n \times 1$ unit vector of ones,

n = number of sectors,

^{T} = transpose of a vector or matrix.

The demand-side model is given by:

(3a) $$X^{D} = AX^{D} + Y,$$

(3b) $$X^{D} = (I - A)^{-1}Y,$$

and the supply-side model is given by:

(4a) $$X^{S} = B^{T}X^{S} + W,$$

(4b) $$X^{S} = (I - B^{T})^{-1}W,$$

where

$A = Z\hat{X}^{-1}$, $n \times n$ matrix of direct-input coefficients, a_{ij},

$B = \hat{X}^{-1}Z$, $n \times n$ matrix of direct-output coefficients, b_{ij},

I = $n \times n$ identity matrix,

$\hat{}$ transforms a vector into a diagonal matrix,

^{D} stands for the demand-side model,

^{S} stands for the supply-side model.

The demand-side model incorporates the assumption of fixed technology, with causality running from final demand to output, and the supply-side model incorporates the assumption of fixed allocation, with causality running from value added to output. Both models generate the output of the base year when fed that year's final demand or value added vectors. The solutions to Equations (3b) and (4b) will diverge when projected exogenous final demand or value added vectors are used. The conventional models can respectively be expressed by:

(5) $$\{ X^* = (I - A^*)^{-1}Y^* : \quad Y^* = \overline{Y};\ A^*\hat{X}^* = A\hat{X}^* \},$$

(6) $$\{ X^* = (I - B^{*T})^{-1}W^* : \quad W^* = \overline{W};\ \hat{X}^*B^* = \hat{X}^*B \},$$

where

$*$ denotes a projected vector or matrix,

$-$ denotes an exogenously given variable.

The constraints on the coefficient matrices given in Equations (5) and (6) are equivalent to $A^* = A$ and $B^* = B$ respectively. This connection between the base year and projected coefficient matrices through the projected interindustry flow matrix constitutes the functional basis of the generalized input-output model.

CHAPTER 2

NOMINAL GENERALIZED INPUT-OUTPUT MODEL

Equations (3a) and (4a) can be written as the following set of equations:

(7) $$x_i^{D*} = \Sigma_{j=1}^{n} a_{ij}^{*} x_i^{D*} + y_i^{*}, \qquad (i=1,\ldots,n),$$

(8) $$x_i^{S*} = \Sigma_{j=1}^{n} b_{ji}^{*} x_i^{S*} + w_i^{*}, \qquad (i=1,\ldots,n).$$

The generalized input-output model combines these sets of equations by taking their convex combination and constraining the otherwise divergent output vectors X^{D*} and X^{S*} to be equal.

(9) $$x_i^{*} = \Sigma_{j=1}^{n} (\lambda_i a_{ij}^{*} + (1-\lambda_i) b_{ji}^{*}) x_i^{*} + \lambda_i y_i^{*} + (1-\lambda_i) w_i^{*}, \quad (i=1,\ldots,n),$$

which can be written in the following matrix form:

$$\Lambda A^{*} X^{*} + \Lambda Y^{*} + (I-\Lambda) B^{*T} X^{*} + (I-\Lambda) W^{*} = X^{*},$$

$$(\Lambda A^{*} + (I-\Lambda) B^{*T}) X^{*} + (\Lambda Y^{*} + (I-\Lambda) W^{*}) = X^{*},$$

(10) $$X^{*} = T^{*} X^{*} + S^{*},$$

where

$T^{*} = \Lambda A^{*} + (I-\Lambda) B^{*T}$, $n \times n$ generalized direct coefficient matrix,

$S^{*} = \Lambda Y^{*} + (I-\Lambda) W^{*}$, $n \times 1$ generalized exogenous vector,

Λ is a diagonal matrix whose main diagonal contains the sectoral demand-supply mix parameters λ_i, where $0 \le \lambda_i \le 1$, $i=1,\ldots,n$.

In order to solve Equation (10) for X^{*}, we need to postulate a relationship linking A^{*} and B^{*} to A and B of the base year.[1] This additional relationship that closes the system is obtained by taking the generalized interindustry

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[1] A possibility would be to hold the direct coefficient matrix T constant: $T^{*} = T = \Lambda A + (I-\Lambda) B^{T}$. This convex combination has been suggested by Oosterhaven (1981) in another context. However, it proves to be unstable, as is shown in Appendix 2.

flow matrix to be a similar convex combination of the interindustry flow matrices of the two models and constraining them to be the same:

$$Z^* = \Lambda Z^{D*} + (I-\Lambda)Z^{S*},$$

$$Z^* = \Lambda A\hat{X}^* + (I-\Lambda)\hat{X}^*B. \quad (11)$$ [1]

The coefficient matrices of the generalized model are expressed by Equations (12) and (13):

$$A^* = Z^*\hat{X}^{*-1},$$

$$A^* = \Lambda A + (I-\Lambda)\hat{X}^*B\hat{X}^{*-1}, \quad (12)$$

$$B^* = \hat{X}^{*-1}Z^*,$$

$$B^* = (I-\Lambda)B + \Lambda\hat{X}^{*-1}A\hat{X}^*. \quad (13)$$

Now, we are ready to solve for the system given in Equation (10). The generalized coefficient matrix T^* can be expressed in terms of the base year coefficient matrices. Substituting Equations (12) and (13) into T^*:

$$T^* = \Lambda A^* + (I-\Lambda)B^{*T},$$

$$T^* = \Lambda(\Lambda A + (I-\Lambda)\hat{X}^*B\hat{X}^{*-1}) + (I-\Lambda)(B^T(I-\Lambda) + \hat{X}^*A^T\hat{X}^{*-1}\Lambda),$$

$$T^* = \Lambda^2A + (I-\Lambda)B^T(I-\Lambda) + \Lambda(I-\Lambda)(\hat{X}^*B\hat{X}^{*-1}) + (I-\Lambda)\hat{X}^*A^T\hat{X}^{*-1}\Lambda,$$

$$T^* = \Lambda^2A + (I-\Lambda)B^T(I-\Lambda) + (I-\Lambda)[\hat{X}^*(\Lambda B + A^T\Lambda)\hat{X}^{*-1}]. \quad (14)$$

The solution to the generalized input-output system can be expressed as an iterative solution:

$$X^* = (I - T^*)^{-1}S^*. \quad (15)$$

The representation of the generalized model analogous to Equations (5) and (6) is:

$$\{ X^* = (I - \Lambda A^* - (I-\Lambda)B^{*T})^{-1}S^* : \; S^* = \bar{S};\; \Lambda = \bar{\Lambda}; \quad (16)$$
$$\Lambda A^*\hat{X}^* + (I-\Lambda)\hat{X}^*B^* = \Lambda A\hat{X}^* + (I-\Lambda)\hat{X}^*B \}.$$

---

[1] Note that by definition $Z^* = \Lambda A^*\hat{X}^* + (I-\Lambda)\hat{X}^*B^*$.

The difference between Equations (3b) and (4b) and Equation (15) is the presence of the endogenous variable X^* in the generalized inverse matrix which implies that Equation (15) must be solved iteratively. The feasibility of the system given by Equation (15) depends on the existence and uniqueness of the convergence of the iteration. The system must be able to generate a unique nonnegative set $\{X^*, A^*, B^*\}$ that satisfies the solution and constraint expressed in (16).[1]

REDUCED FORM

In this section, the reduced form of the generalized input-output system is presented. The algebraic manipulation eliminates X^* from the right-hand side of Equation (15) and reduces the inverse matrix to a function of the base year variables only. The reduced form, given by Equation (17), generates the identical solution as Equation (15). Going back to Equation (10):

$$X^* = (\Lambda^2 A + (I-\Lambda)B^T(I-\Lambda) + (I-\Lambda)[\hat{X}^*(\Lambda B + A^T\Lambda)\hat{X}^{*-1}])\, X^* + S^*,$$

$$X^* = [\Lambda^2 A + (I-\Lambda)B^T(I-\Lambda)]X^* + (I-\Lambda)\hat{X}^*(\Lambda B + A^T\Lambda)i + S^*,$$

$$X^* = [\Lambda^2 A + (I-\Lambda)B^T(I-\Lambda)]X^* + (I-\Lambda)\widehat{[(\Lambda B + A^T\Lambda)i]}X^* + S^*,$$

$$X^* = T_r X^* + S^*,$$

$$X^* = (I - T_r)^{-1} S^*, \tag{17}$$

where

$$T_r = \Lambda^2 A + (I-\Lambda)B^T(I-\Lambda) + (I-\Lambda)\widehat{[(\Lambda B + A^T\Lambda)i]}.$$

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[1] Similar to the constraints in Equations (5) and (6), the constraint here relates the projected and base year coefficient matrices through the construction of the projected interindustry flow matrix, Z^*.

The fact that Equations (15) and (17) transform S^* identically into X^*, permits us to seek the convergence of Equation (15) in the properties of $(I - T_r)$. Before exploring the issue of stability further, however, we will discuss certain aspects of the "consistency" of the model. The notion of consistency pertains here to the condition that the projected table of an input-output model must be balanced, that is, it must satisfy the identities (1) and (2).

CONSISTENCY

In the demand-side (supply-side) model, value added (final demand) is determined endogenously. For the demand-side model:

$$W^* = \hat{l}(I - A)^{-1}Y^*, \tag{18}$$

and for the supply-side model:

$$Y^* = \hat{d}(I - B^T)^{-1}W^*, \tag{19}$$

where

l = row vector of value added coefficients,

d = row vector of final demand coefficients.

The equivalent row vector of coefficients for the generalized model, t, can be expressed as:

$$t^* = l^*\Lambda + d^*(I-\Lambda). \tag{20}$$

We have:

$$\hat{t}^*X^* = (\Lambda\hat{l}^* + (I-\Lambda)d^*)\hat{X}^*,$$

$$\hat{t}^*X^* = \Lambda\hat{l}^*X^* + (I-\Lambda)d^*\hat{X}^*.$$

Substituting and switching, we construct a relationship between Y^* and W^*, analogous to Equations (18) and (19):

$$\Lambda W^* + (I-\Lambda)Y^* = \hat{t}^*(I - T^*)^{-1}(\Lambda Y^* + (I-\Lambda)W^*). \tag{21}$$

Define the endogenous variable for the generalized model as:

$$R^* = \Lambda W^* + (I-\Lambda)Y^*. \tag{22}$$

Then, according to Equation (21), consistency requires that:

$$R^* = \hat{t}^*(I - T^*)^{-1}S^*. \tag{23}$$

Although in the original models, the final demand and value added vectors are unambiguously either exogenous or endogenous, in the generalized model they are both exogenous and endogenous at the same time. In fact R^* and S^* are "mirror images" of each other, and the extent of exogeneity of final demand (value added) for a given sector is an increasing (decreasing) function of the value of that sector's λ.

Rearranging Equation (21) and solving for W^*:

$$W^* = F^*Y^*, \tag{24}$$

where

$$F^* = [(\Lambda - \hat{t}^*(I - T^*)^{-1}(I-\Lambda)]^{-1} \; [\hat{t}^*(I - T^*)^{-1}\Lambda - (I-\Lambda)] \;.$$

However, Equation (24) gives a relation over the variables of the projected table. Fortunately, we can eliminate the a priori unknown generalized direct coefficient matrix, T^*, and row vector of coefficients, t^*, from the matrix that gives the relationship between the feasible pairs of projected final demand and value added vectors, F^*. Following from definition (20):

$$t^* = i^T\Lambda - i^TA^*\Lambda + i^T(I-\Lambda) - i^TB^{*T}(I-\Lambda),$$ [1]

$$t^* = i^T - i^T[A^*\Lambda + B^{*T}(I-\Lambda)].$$

Substituting Equations (12) and (13):

$$t^* = i^T[I - \Lambda A\Lambda - B^T(I-\Lambda)^2] - i^T[\hat{X}^*((I-\Lambda)B\Lambda + A^T\Lambda(I-\Lambda))\hat{X}^{*-1}],$$

---

[1] Note that a balanced interindustry table requires that $i^TA^* + l^* = i^T$ and $i^TB^{*T} + d^* = i^T$.

$$t^*\hat{X}^* = i^T[I - \Lambda A\Lambda - B^T(I-\Lambda)^2]\hat{X}^* - i^T\hat{X}^*[(I-\Lambda)B\Lambda + A^T\Lambda(I-\Lambda)],$$

$$X^{*T}\hat{t}^* = X^{*T}\widehat{[i^T(I - \Lambda A\Lambda - B^T(I-\Lambda)^2)]} - X^{*T}[(I-\Lambda)B\Lambda + A^T\Lambda(I-\Lambda)].$$

Transposing:

$$\hat{t}^*X^* = \widehat{[i^T(I - \Lambda A\Lambda - B^T(I-\Lambda)^2)]}X^* - [\Lambda B^T(I-\Lambda) + \Lambda(I-\Lambda)A]X^*,$$

$$\hat{t}^*X^* = [\widehat{[i^T(I - \Lambda A\Lambda - B^T(I-\Lambda)^2)]} - [\Lambda B^T(I-\Lambda) + \Lambda(I-\Lambda)A]]X^*.$$

Finally, we can construct the reduced form of Equation (23):

$$\text{(25)} \quad R^* = [\widehat{[i^T(I - \Lambda A\Lambda - B^T(I-\Lambda)^2)]} - [\Lambda B^T(I-\Lambda) + \Lambda(I-\Lambda)A]](I - T_r)^{-1}S^*.$$

Substituting into R^* and S^*, and rearranging to solve for W^*, the reduced form of Equation (24) becomes:

$$\text{(26)} \qquad W^* = F_rY^*,$$

where

$$F_r = [\Lambda - K(I-\Lambda)]^{-1}\,[K\Lambda - (I-\Lambda)],$$

$$K = [\widehat{[i^T(I - \Lambda A\Lambda - B^T(I-\Lambda)^2)]} - [\Lambda B^T(I-\Lambda) + \Lambda(I-\Lambda)A]](I - T_r)^{-1}.$$

The model translates the exogenous variable $S^* = \overline{S}$ into the projected final demand and value added vectors by means of Equations (27) and (28):

$$\text{(27)} \qquad W^* = [\Lambda F_r^{-1} + (I-\Lambda)]^{-1}S^*,$$

$$\text{(28)} \qquad Y^* = [\Lambda + (I-\Lambda)F_r]^{-1}S^*.$$ [1]

Hence, S^* must lie in a cone that is more restricted than the positive orthant, such that the above two equations produce nonnegative vectors Y^* and W^*. It is possible to attach even further restrictions on S^* to set minimum levels and proportions to these vectors. The implication is that, in projecting the table, the analyst does not have control over the levels of both Y^* and W^*. Only one of them or their convex combination can be set

[1] For any $\lambda_i = 0.5$, Equations (26)-(28) involve singular matrices and the relations cannot be formed directly. However, for all equations, we can take the limit as λ_i goes to 0.5 from either side since the relation is continuous.

exogenously. If the generalized exogenous vector S^* is formed by the convex combination of two arbitrary final demand and value added vectors, the model will generate a different pair of $\{Y^*, W^*\}$ unless the original choices happen to satisfy Equation (26).

In this section, we have demonstrated the implicit constraints of the generalized input-output model such that it is internally consistent. This means that the model generates a balanced projected table with nonnegative final demand and value added vectors. In the next section, we investigate the question of the nonnegativity of the generalized inverse matrix and output projections.

STABILITY

If the generalized input-output model generates a nonnegative output vector given any convex combination of any nonnegative final demand and value added vectors, then it is said to be economically feasible, or stable. Stability of the generalized input-output model, in this sense, hinges on the nonsingularity and inverse-positivity of the matrix $(I - T_r)$. In order to investigate the conditions under which this holds, we introduce several definitions and well-known theorems of linear algebra on nonnegative matrices.

DEFINITION 1. $R^{n \times n}$ is the space of $n \times n$ real matrices. $Z^{n \times n}$ is the space of matrices such that $\{ G = (g_{ij}) \in R^{n \times n};\ g_{ij} \leq 0,\ i \neq j;\ g_{ij} \geq 0,\ i=j \}$.

DEFINITION 2. A real $n \times n$ matrix $H = (h_{ij})$ is called nonnegative (positive) if $h_{ij} \geq 0$ ($h_{ij} > 0$) for $i,j=1,\ldots,n$; if H is nonnegative (positive), it is denoted by $H \geq 0$ ($H > 0$).

DEFINITION 3. Let $H = (h_{ij})$ be a real $n \times n$ matrix with eigenvalues ξ_i, $i=1,\ldots,n$; then $\rho(H) \equiv \max |\xi_i|$ $(i=1,\ldots,n)$, is the spectral radius, or equivalently, the dominant eigenvalue of H.

DEFINITION 4. A real $n \times n$ matrix $G = (g_{ij}) \in Z^{n\times n}$ can be expressed in the form:

$$G = sI - H, \quad s > 0,\ H \geq 0;$$

any matrix of this form for which $s \geq \rho(H)$, is called an M-matrix, and when $s > \rho(H)$, it is called a nonsingular M-matrix.

LEMMA 1. Let H be a $n \times n$ nonnegative matrix. Then the conditions (I)-(IV) below are equivalent and imply each other:

(I) $\Sigma_{k=0}^{\infty} H^k$ converges.

(II) The spectral radius of H is less than one in absolute value.

(III) $(I - H)^{-1}$ exists and is nonnegative.

(IV) For any nonnegative vector S, the equation $(I - H)X = S$ has a unique nonnegative solution. (H is productive and the input-output model is economically feasible.)[1]

---

[1] For a discussion of this lemma, see Miyazawa (1976, p.15).

For the demand-side (supply-side) model, we can prove stability by showing that the spectral radius of A (B) is less than one because all the column (row) sums of the coefficient matrix are less than one given a positive final demand (value added) vector for the base year.[1] In the case of the reduced form of the generalized model, T_r might have some column (or row) sums totalling more than unity. However, this does not imply that $\rho(T_r)$ is not less than one and our strategy will be to prove that $\rho(T_r)$ is actually less than one.

THEOREM 1. For the input-output model given by $G \equiv (I - H) \in Z^{n \times n}$, the model is feasible iff G is a nonsingular M-matrix.

Proof : By Definition 4, if G is a nonsingular M-matrix, then $\rho(H) < 1$ and by Lemma 1(II) and 1(IV), the model is feasible. If G is not a nonsingular M-matrix, then $\rho(H) \geq 1$, and by the same lemma, the model is not feasible.

THEOREM 2. For $G \in Z^{n \times n}$, each of the following conditions are equivalent to the statement "G is a nonsingular M-matrix."

(I) All principal minors of G are positive.

(II) G is inverse-positive; that is, G^{-1} exists and $G^{-1} \geq 0$.

---

[1] In the case where some column (row) sums equal one and others are less than one, feasibility depends on the irreducibility of the system where no permutations will result in an upper left submatrix with column (row) sums equalling one. This is what Solow (1952) refers to as the coupling of sectors. Our discussion here will be limited to irreducible coefficient matrices.

(III) There exists a positive diagonal matrix D such that GD has all positive row sums.

The first one is the well-known Hawkins-Simon condition for the feasibility of the open input-output model. The second condition connects Theorem 2 to Lemma 1(III) and through its equivalence to Lemma 1(II), to the definition of an M-matrix. We will use the third condition to prove, through this linkage, that $(I - T_r)$ is a nonsingular M-matrix.[1]

THEOREM 3. For a generalized input-output model given by $X^* = (I - T_r)^{-1}S^*$, the model is feasible if the final demand and value added vectors of the base year are positive; that is, if the column sums of both A and B^T are less than one.

Proof : Let $D = \hat{X}$ of Theorem 2(III). The row sums of $(I - T_r)\hat{X}$ are given by:

$$
\begin{aligned}
(I - T_r)\hat{X}i &= [I - \Lambda^2A - (I-\Lambda)B^T(I-\Lambda) - (I-\Lambda)\widehat{[i^T(\Lambda A + B^T\Lambda)]}]\hat{X}i. \\
&= \hat{X}i - \Lambda^2A\hat{X}i - (I-\Lambda)B^T\hat{X}(I-\Lambda)i - (I-\Lambda)[\widehat{(i^T\Lambda A)}\hat{X} + \widehat{(i^TB^T\Lambda)}\hat{X}]i, \\
&= X - \Lambda^2Zi - (I-\Lambda)Z^T(I-\Lambda)i - (I-\Lambda)\widehat{[(\Lambda Z + Z^T\Lambda)i]}i, \\
&= X - \Lambda^2Zi - (I-\Lambda)Z^T(I-\Lambda)i - (I-\Lambda)[\Lambda Zi + Z^T\Lambda i], \\
&= X - \Lambda Zi - (I-\Lambda)Z^Ti, \\
&= X - \Lambda(X - Y) - (I-\Lambda)(X - W), \\
&= \Lambda Y + (I-\Lambda)W, \\
&> 0.
\end{aligned}
$$

[1] Berman and Plemmons (1979) lists 50 such equivalent conditions. For the proof of Theorem 2, see p.138 of that book.

Hence, by Theorem 2, $(I - T_r)$ is a nonsingular M-matrix, and by Theorem 1, the generalized input-output system is feasible. Once the model generates X^*, the interindustry flow matrix Z^* can be formed through Equation (11) and is clearly nonnegative. Therefore, for any exogenous vector S^* defined by the cone:

$$\{ S^* : \ [\Lambda F_r^{-1} + (I-\Lambda)]^{-1}S^* \geq 0; \ [\Lambda + (I-\Lambda)F_r]^{-1}S^* \geq 0 \},$$

the generalized input-output model projects a nonnegative and balanced input-output table.

HORIZONTAL VERSUS VERTICAL COUPLING

The closure of the generalized model given by Equation (11) carries an implicit assumption about the manner technology and allocation change over a projection. Let us assume a 4-sector model where the first two sectors are demand-driven ($\lambda_1 = \lambda_2 = 1$) and the last two sectors are supply-driven ($\lambda_3 = \lambda_4 = 0$). Denoting cell-by-cell matrix division by ./, the projected coefficient matrices will differ from the base year as shown below:

$$A^* ./A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ ? & ? & 1 & ? \\ ? & ? & ? & 1 \end{bmatrix}$$

$$B^* ./B = \begin{bmatrix} 1 & ? & ? & ? \\ ? & 1 & ? & ? \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

If a sector is demand-driven, then the input of that sector to all sectors is a fixed proportion of the output of the latter sectors.[1] If a sector is supply-driven, then the output of that sector is allocated to all sectors in fixed proportion to the output of that sector; in other words, its "allocation" is fixed. We will call this "horizontal coupling" due to the dominance of the horizontal structure over the vertical one. This implies that demand- or supply-drivenness operates over the notion of "commodity/markets" rather than "industry/technology." In other words, it is the "commodity" that is demand- or supply-driven rather than the "industry." This horizontal structure corresponds to quantity rationing of intermediate inputs which are in excess demand.

However, Equation (11) is not the only way to close the generalized system. Replacing it by a right-hand side convex combination of the demand- and supply-side interindustry flows:

$$Z^* = A\hat{X}^*\Lambda + \hat{X}^*B(I-\Lambda), \tag{29}$$

The coefficient matrices of the new generalized model are given by:

$$A^* = Z^*\hat{X}^{*-1},$$

$$A^* = A\Lambda + \hat{X}^*B\hat{X}^{*-1}(I-\Lambda), \tag{30}$$

$$B^* = \hat{X}^{*-1}Z^*,$$

$$B^* = B(I-\Lambda) + \hat{X}^{*-1}A\hat{X}^*\Lambda. \tag{31}$$

Substituting Equations (30) and (31) into T^*:

$$T^* = \Lambda A^* + (I-\Lambda)B^{*T},$$

$$T^* = \Lambda(A\Lambda + \hat{X}^*B\hat{X}^{*-1}(I-\Lambda)) + (I-\Lambda)((I-\Lambda)B^T + \Lambda\hat{X}^*A^T\hat{X}^{*-1}),$$

$$T^* = \Lambda A\Lambda + (I-\Lambda)^2B^T + \Lambda\hat{X}^*B\hat{X}^{*-1}(I-\Lambda) + (I-\Lambda)\Lambda\hat{X}^*A^T\hat{X}^{*-1},$$

[1] However, the "technology" of the demand-driven sector is not totally fixed.

$$(32) \qquad T^* = \Lambda A \Lambda + (I-\Lambda)^2 B^T + \Lambda[\hat{X}^*(B(I-\Lambda) + (I-\Lambda)A^T)\hat{X}^{*-1}].$$

The model is expressed by:

$$(33) \qquad \{\, X^* = (I - \Lambda A^* - (I-\Lambda)B^{*T})^{-1}S^* \;:\; S^* = \bar{S};\ \Lambda = \bar{\Lambda};$$
$$A^*\hat{X}^*\Lambda + \hat{X}^*B^*(I-\Lambda) = A\hat{X}^*\Lambda + \hat{X}^*B(I-\Lambda) \,\}.$$

The reduced form of the model is given by:

$$(34) \qquad X^* = (I - T_r)^{-1}S^*,$$

where

$$T_r = \Lambda A \Lambda + (I-\Lambda)^2 B^T + \Lambda[\widehat{(B(I-\Lambda) + (I-\Lambda)A^T)i}].$$ [1]

Using the same reasoning as for the proof of stability of the horizontal generalized model, we can show that $(I - T_r)$ of Equation (34) is a nonsingular M-matrix. Let $D = \hat{X}$ of Theorem 2(III):

$$
\begin{aligned}
(I - T_r)\hat{X}i &= [I - \Lambda A \Lambda - (I-\Lambda)^2 B^T - \Lambda[\widehat{(B(I-\Lambda) + (I-\Lambda)A^T)i}]]\hat{X}i. \\
&= \hat{X}i - \Lambda A\hat{X}\Lambda i - (I-\Lambda)^2 B^T\hat{X}i - \Lambda[\widehat{(i^T(I-\Lambda)B^T)}\hat{X} + \widehat{(i^T A(I-\Lambda))}\hat{X}]i, \\
&= X - \Lambda Z \Lambda i - (I-\Lambda)^2 Z^T i - \Lambda[\widehat{((I-\Lambda)Z^T + Z(I-\Lambda))i}]i, \\
&= X - \Lambda Z \Lambda i - (I-\Lambda)^2 Z^T i - \Lambda[(I-\Lambda)Z^T i + Z(I-\Lambda)i], \\
&= X - \Lambda Z i - (I-\Lambda)Z^T i, \\
&= X - \Lambda(X - Y) - (I-\Lambda)(X - W), \\
&= \Lambda Y + (I-\Lambda)W, \\
&> 0.
\end{aligned}
$$

Hence, the vertical generalized model is also stable.

The transformation of Y^* into W^* is different for this model and the reduced form of F^* is given by:

---

[1] The derivations of the reduced form matrices T_r and F_r for this model are similar to those of the horizontal generalized model and are not carried out here. Furthermore, the transformations of S^* into Y^* and W^* are given by the Equations (27) and (28) of the horizontal generalized model, but, of course, are different because the matrices T_r and F_r are different for the two versions.

$$F_r = [\Lambda - K(I-\Lambda)]^{-1}\,[K\Lambda - (I-\Lambda)],$$

where

$$K = [\overline{[i^T(I - A\Lambda^2 - (I-\Lambda)B^T(I-\Lambda))]} - [\Lambda(I-\Lambda)B^T + (I-\Lambda)A\Lambda]](I - T_r)^{-1}.$$

For the same example, the pattern of change of the coefficient matrices over a projection will look like:

$$A^* ./A \quad = \quad \begin{bmatrix} 1 & 1 & ? & ? \\ 1 & 1 & ? & ? \\ 1 & 1 & 1 & ? \\ 1 & 1 & ? & 1 \end{bmatrix}$$

$$B^* ./B \quad = \quad \begin{bmatrix} 1 & ? & 1 & 1 \\ ? & 1 & 1 & 1 \\ ? & ? & 1 & 1 \\ ? & ? & 1 & 1 \end{bmatrix}$$

Here, if a sector is demand-driven, then the input of all other sectors into that sector is a fixed proportion of the former sector's output. If a sector is supply-driven, then the output of each sector's output is a fixed proportion of the latter's output. In other words, the technology of a demand-driven sector is fixed, but the allocation of the output of a supply-driven sector is not totally fixed. The structure displayed here is a transposition of the structure of the horizontal generalized model and will be called "vertical coupling." In the vertical generalized model, the notion of demand- or supply-drivenness operates over the notion of "industry/technology."

ESTIMATION OF DEMAND-SUPPLY MIX PARAMETERS

In this section, we estimate the sectoral demand-supply mix parameters between two given benchmark years for both versions of the generalized input-ouput model. The methodology used minimizes the sum of absolute deviations between the projected and actual output vectors over the feasible values of λ_i's, which are between zero and one.[1] The selection of this index over another one (e.g., the sum of absolute deviations between the projected and actual interindustry flows) is clearly arbitrary. Furthermore, the methodology is crude because we assume that demand- or supply-drivenness is the only cause of variation in sectoral output. On the other hand, demand- or supply-drivenness itself is subject to variation as it probably depends on the business cycle as well as exogenous shocks. Subsequently, the results in Tables 2.1 and 2.2 are obscured by the fact that the cycle was at a different point of its fluctuation for each of the benchmark years.[2] Despite these shortcomings, however, this exercise should be taken as a preliminary attempt to estimate the demand-supply mix parameters that might indicate their magnitude and stability. The data used are the 7-sector United States input-output tables for the benchmark years 1947, 1958, 1963, 1967, 1972,

[1] In projecting the output of the latter year, we use the coefficient matrices of the former year and the final demand and value added vectors of the latter year. Given Equation (15) for the horizontal version and Equation (34) for the vertical version, the estimation problem defined as such becomes a discontinuous nonlinear optimization problem. A grid-search program is created to calculate the results below.

[2] Further research is needed to develop an econometric model to explain the variations for the sectoral demand-supply mix parameters.

and 1977.[1] The methodology is applied to each of the five intervals between these six years. The results are given in Tables 2.1 and 2.2. Also included in the tables are the minimized sum of absolute deviation and the percentage of this to the actual output.

The estimations for the horizontal generalized model in Table 2.1 suggest that the commodities of the agricultural and mining sectors are more supply-driven (the averages being close to zero), those of construction, trade and transportation, and government enterprises are more demand-driven (the averages being close to one), and those of manufacturing, and services fall someplace closer to the middle. We found that, of the five estimations, the variation is highest for trade and transportation, construction, manufacturing, and services, in that order.

The estimations for the vertical generalized model listed in Table 2.2, on the other hand, indicate the demand-supply mix parameters for the industry, rather than the commodity of a sector. For the average, they are within .20 of the results in Table 2.1, except for sector 7. For the estimations of the vertical model, we found that the variation is highest for manufacturing, followed by mining, construction, and services.

---

[1] For the first four benchmark years, data are aggregated from the 23-sector level input-output data published in U.S. Bureau of the Census (1976). For the benchmark years 1972 and 1977, data are aggregated from the 23-sector level Use and Make matrices published in Miller and Blair (1985). (It should be noted here that the methodology of preparing the U.S. tables has changed after and including 1972, and the difference might be reflected in the results.) The values are in millions of current dollars at producer's prices. The 7-sector classification is: 1. Agriculture, 2. Mining, 3. Construction, 4. Manufacturing, 5. Trade and Transportation, 6. Services, 7. Other (this sector consists of government enterprises and scrap and secondhand goods). For a more detailed discussion of the data, refer to Appendix 1.

Table 2.1. Estimated Demand-Supply Mix Parameters for the Horizontal Model

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	$\Sigma\|X^*-X\|$	% of ΣX
1947/58	.02	.00	.00	1.00	1.00	.18	1.00	2.2×10^4	2.70
1958/63	.27	.06	1.00	.48	.00	.68	.74	$.6 \times 10^4$	.57
1963/67	.07	.02	.87	1.00	1.00	.04	1.00	1.2×10^4	.83
1967/72	.46	.27	.57	.08	.46	.51	1.00	3.3×10^4	.46
1972/77	.00	.00	1.00	.44	1.00	.94	1.00	9.4×10^4	2.56
Average	.16	.07	.69	.60	.69	.47	.95		1.64

Table 2.2. Estimated Demand-Supply Mix Parameters for the Vertical Model

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	$\Sigma\|X^*-X\|$	% of ΣX
1947/58	.01	.00	.48	.01	.38	.81	.38	1.6×10^4	1.90
1958/63	.39	.03	.97	.27	.70	.83	.15	$.8 \times 10^4$	.76
1963/67	.55	.40	.99	.00	.97	.00	.00	$.8 \times 10^4$	.55
1967/72	.79	.96	.33	.76	.99	.74	.00	1.8×10^4	.84
1972/77	.00	.00	.10	1.00	1.00	.81	.00	8.2×10^4	2.25
Average	.35	.27	.57	.41	.81	.64	.11		1.26

The last column in each table is a measure of the "fit" of the output projections when the models are used with their "optimal" demand-supply mix parameters. The poor fit for the 1947/58 projection might be due to the relatively long time interval between the two benchmark years and the fact that 1947 was a boom year following World War II, while 1958 was a recession year. The poor fit of the 1972/77 projection might be attributed to the effects of the 1973 oil crisis.

COMPARATIVE PERFORMANCE OF THE NOMINAL MODELS

In this section, we use the same 7-sector U.S. input-output tables for the six benchmark years to test the relative predictive powers of the conventional models against the two versions of the generalized model. For each model, the coefficient matrices of the first five benchmark years are combined with the final demand and/or value added vectors of the following benchmark year to project the output (X^*), the interindustry flow (Z^*), the two coefficient matrices (A^* and B^*), and the two coefficient vectors (l^* and d^*) of the latter year. For example, projecting the 1967 input-output table with the coefficient matrices of 1963, the exogenous vectors used are $Y = Y^{67}$ for the demand-side model, $W = W^{67}$ for the supply-side model, and $S = \Lambda Y^{67} + (I-\Lambda)W^{67}$ for the generalized models. For the latter, the sectoral demand-supply mix parameters are assigned the values 0, 0.5, or 1, according to the averages shown in Tables 2.1 and 2.2. The sectoral demand-supply mix parameters assumed for each model are given in Table 2.3.

Table 2.3. Demand-Supply Mix Parameters Assumed for Each Model

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Demand	1	1	1	1	1	1	1
Horizontal	0	0	1	0.5	1	0.5	1
Vertical	0.5	0	0.5	0.5	1	0.5	0
Supply	0	0	0	0	0	0	0

The ex post forecasts are made only for the consecutive years, yielding five projections for the six benchmark years. Table 2.4 shows the mean absolute cell-by-cell difference for the six projected vectors/matrices against their actual values.[1] The average for mean absolute deviation for the output and interindustry flow measures in Table 2.4 weigh each projection's mean absolute deviation the same. However, in nominal values, total gross output has increased more than four times from 1958 to 1977. Table 2.5 gives total absolute deviation as a percentage of total gross output in order to make the comparison among the five projections more meaningful.[2] On the other hand, the measure of mean absolute deviation does not detect large percentage differences that are relatively small in magnitude. This, however, is captured by the mean percentage deviation, which is calculated only for the output projections and is shown in Table 2.6.[3] (Large percentage differences for some cells that are low in magnitude make this measure meaningless for the other categories.) Finally, the median percentage deviation is given in Table 2.7.[4]

For the output projections, the horizontal model performs better than both demand- and supply-side models, for all projections and under all

---

[1] For the 1947/58 projection, this measure is $(1/n)\Sigma|X^{*}-X^{58}|$ for the output, $(1/n^{2})\Sigma|Z^{*}-Z^{58}|$ for the interindustry flow etc.

[2] For the 1947/58 projection, this measure is $100(\Sigma|X^{*}-X^{58}|)/(\Sigma X^{58})$ for the output, and $100(\Sigma|Z^{*}-Z^{58}|)/(\Sigma X^{58})$ for the interindustry flow.

[3] For the 1947/58 projection, this measure is $(1/n)\Sigma 100|i-(X^{*}./X^{58})|$ for the output.

[4] For the 1947/58 projection, this measure is Median $(100|i-(X^{*}./X^{58})|)$ for the output, Median $(100|ii^{T}-(Z^{*}./Z^{58})|)$ for the interindustry flow matrix etc. The significance of this index, however, is not as clear as the other ones, and the results of Table 2.7 are downplayed.

Table 2.4. Mean Absolute Deviation (Case 1)

OUTPUT

	Demand	Supply	Horizontal	Vertical
1947/58	8703	6518	5116	5982
1958/63	4033	2923	1657	2680
1963/67	4577	3243	2031	3533
1967/72	20429	35671	10721	7786
1972/77	27690	18829	14080	18194
Average	13086	13437	6721	7635

INTERINDUSTRY FLOW

	Demand	Supply	Horizontal	Vertical
1947/58	1685	1440	1238	1551
1958/63	1057	932	902	931
1963/67	1033	882	835	967
1967/72	3525	5429	2641	2582
1972/77	4498	3574	3347	3907
Average	2360	2451	1793	1988

DIRECT-INPUT COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	0.0110	0.0109	0.0103	0.0115
1958/63	0.0079	0.0070	0.0071	0.0071
1963/67	0.0055	0.0058	0.0056	0.0059
1967/72	0.0122	0.0123	0.0108	0.0079
1972/77	0.0081	0.0069	0.0068	0.0074
Average	0.0089	0.0086	0.0081	0.0080

DIRECT-OUTPUT COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	0.0153	0.0186	0.0156	0.0190
1958/63	0.0071	0.0066	0.0063	0.0078
1963/67	0.0046	0.0050	0.0052	0.0052
1967/72	0.0107	0.0233	0.0109	0.0137
1972/77	0.0248	0.0119	0.0113	0.0140
Average	0.0125	0.0131	0.0099	0.0119

VALUE ADDED COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	0.0267	0.0488	0.0395	0.0361
1958/63	0.0239	0.0128	0.0154	0.0111
1963/67	0.0133	0.0163	0.0132	0.0143
1967/72	0.0695	0.0536	0.0532	0.0304
1972/77	0.0232	0.0230	0.0158	0.0234
Average	0.0313	0.0309	0.0274	0.0231

FINAL DEMAND COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	0.0240	0.0798	0.0314	0.0789
1958/63	0.0116	0.0127	0.0100	0.0204
1963/67	0.0097	0.0081	0.0098	0.0155
1967/72	0.0362	0.1394	0.0483	0.0723
1972/77	0.1526	0.0708	0.0666	0.0880
Average	0.0468	0.0622	0.0332	0.0550

Table 2.5. Total Absolute Deviation as Percentage of Total Output (Case 1)

OUTPUT

	Demand	Supply	Horizontal	Vertical
1947/58	7.34	5.50	4.31	5.04
1958/63	2.65	1.92	1.09	1.76
1963/67	2.27	1.61	1.01	1.76
1967/72	6.80	11.88	3.57	2.59
1972/77	5.29	3.60	2.69	3.48
Average	4.87	4.90	2.53	2.93

INTERINDUSTRY FLOW

	Demand	Supply	Horizontal	Vertical
1947/58	9.95	8.50	7.31	9.15
1958/63	4.87	4.29	4.16	4.29
1963/67	3.59	3.07	2.91	3.36
1967/72	8.22	12.65	6.16	6.02
1972/77	6.02	4.78	4.48	5.22
Average	6.53	6.66	5.00	5.61

Table 2.6. Mean Percentage Deviation (Case 1)

OUTPUT

	Demand	Supply	Horizontal	Vertical
1947/58	11.17	9.70	3.79	8.49
1958/63	6.00	2.47	1.65	2.24
1963/67	4.88	3.38	1.52	3.00
1967/72	7.53	10.09	4.69	3.23
1972/77	14.60	3.79	2.51	4.80
Average	8.84	5.89	2.83	4.35

Table 2.7. Median Percentage Deviation (Case 1)

OUTPUT

	Demand	Supply	Horizontal	Vertical
1947/58	3.88	7.01	2.72	3.45
1958/63	2.86	3.54	1.49	2.11
1963/67	2.39	3.67	1.50	3.34
1967/72	8.36	12.43	3.84	2.22
1972/77	4.60	3.62	2.84	3.93
Average	4.42	6.05	2.48	3.01

INTERINDUSTRY FLOW

	Demand	Supply	Horizontal	Vertical
1947/58	21.2	27.7	21.4	25.3
1958/63	12.1	13.0	11.1	12.3
1963/67	13.6	11.3	8.8	11.4
1967/72	13.0	24.9	17.6	13.3
1972/77	19.6	24.4	18.9	22.5
Average	15.9	20.3	15.6	17.0

DIRECT-INPUT COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	21.1	25.8	21.1	21.7
1958/63	11.3	12.1	11.1	11.9
1963/67	11.9	10.3	9.3	9.1
1967/72	13.0	20.3	18.7	11.9
1972/77	15.4	24.0	15.8	22.0
Average	14.5	18.5	15.2	15.3

DIRECT-OUTPUT COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	21.1	25.7	18.7	23.6
1958/63	14.0	11.7	10.0	10.8
1963/67	11.2	11.4	9.3	10.7
1967/72	12.0	19.8	17.2	17.8
1972/77	20.9	23.5	18.5	21.5
Average	15.8	18.4	14.7	16.9

VALUE ADDED COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	2.86	7.53	4.13	2.93
1958/63	4.06	2.05	2.30	1.47
1963/67	1.88	3.52	1.65	1.86
1967/72	4.14	11.06	3.70	4.03
1972/77	3.40	3.50	1.93	1.54
Average	3.27	5.53	2.74	2.37

FINAL DEMAND COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1947/58	3.73	11.72	5.21	8.29
1958/63	2.94	2.96	2.62	1.77
1963/67	2.33	2.30	2.74	1.73
1967/72	7.71	4.68	5.96	1.34
1972/77	4.82	4.27	4.12	4.82
Average	4.31	5.19	4.13	3.59

indices. The vertical model performs better than either model for at least 4 out of 5 output projections under all indices. Compared with the vertical model, the horizontal model generates better output projections, except for the 1967/72 projection.

For the interindustry flow projections, both generalized models outperform the demand-side model for the index of the mean absolute deviation. Although, for the same index, the horizontal model performs better than the supply-side model for all projections, the vertical model does worse than the supply-side model 3 out of 5 times. The comparison of the index of median percentage deviation for the interindustry flow projection is less clear, with decreasing levels of performance on average by horizontal, demand-side, vertical, and supply-side models.

It is interesting to note that, while on average the supply-side model predicts the direct-input (technology) coefficient matrix better than the demand-side model, the latter predicts the direct-output (allocation) coefficient matrix better than the former. The generalized models, on average, do better than the other two for both comparisons, while the horizontal model outperforms the vertical model in 4 out of 5 projections for the index of mean absolute deviation for both coefficient matrices.

For the value added coefficient vector, on average, the best projection is by the vertical model under both indices, followed by the horizontal model. For the final demand coefficient vector, on average, the horizontal model performs better than both conventional models under both indices, while the vertical model performs the best under the index of median percentage deviation.

All in all, the results in Tables 2.4 - 2.7 suggest that, given their respective demand-supply mix parameters in Table 2.3, the generalized models perform better than the conventional models for the U.S. data. The horizontal model tends to do better than the vertical model in projecting the output, interindustry flow, and coefficient matrices. The difference between the predictive powers of the demand- and supply-side models are less clear in Tables 2.4 - 2.6, while the median percentage deviation index of Table 2.7 favors the demand-side model.

The superior performance of the two generalized models might be attributed to the fact that the demand-supply mix parameters used in Tables 2.4 - 2.7 were not known a priori, but are derived from the data itself after the "future" was known. How would the results differ if we just used those parameters derived from the last period?[1] The results for the two generalized models are given in Tables 2.8 - 2.10. The results for the demand- and supply-side models are repeated for ease of comparison along with their new averages for the four projections. (We omit the median percentage calculations for this round of comparisons.)

The averages for the total absolute output deviation as percentage of total output have increased for both generalized models, while they are still significantly lower than the supply- and demand-side models. For the mean percentage deviation of output, however, the supply-side model now performs better than the vertical model, while the horizontal model has the best average. The horizontal model still has the lowest index for

---

[1] For the 1972/77 projection, for example, we use the 1967/72 estimation, which is given in Tables 2.1 and 2.2 for the horizontal and vertical generalized models, respectively. We omit the 1947/58 projection for obvious reasons.

Table 2.8. Mean Absolute Deviation (Case 2)

OUTPUT

	Demand	Supply	Horizontal	Vertical
1958/63	4033	2923	3498	2239
1963/67	4577	3243	3158	2864
1967/72	20429	35671	19602	22844
1972/77	27690	18829	21618	28073
Average	14182	15167	11969	14005

INTERINDUSTRY FLOW

	Demand	Supply	Horizontal	Vertical
1958/63	1057	932	888	959
1963/67	1033	882	894	865
1967/72	3525	5429	3530	4112
1972/77	4498	3574	4003	4555
Average	2528	2704	2329	2623

DIRECT-INPUT COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1958/63	0.0079	0.0070	0.0070	0.0073
1963/67	0.0055	0.0058	0.0056	0.0056
1967/72	0.0122	0.0123	0.0112	0.0091
1972/77	0.0081	0.0069	0.0077	0.0085
Average	0.0084	0.0080	0.0079	0.0076

DIRECT-OUTPUT COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1958/63	0.0071	0.0066	0.0062	0.0068
1963/67	0.0046	0.0050	0.0050	0.0048
1967/72	0.0107	0.0233	0.0110	0.0192
1972/77	0.0248	0.0119	0.0123	0.0394
Average	0.0118	0.0117	0.0086	0.0176

VALUE ADDED COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1958/63	0.0239	0.0128	0.0138	0.0147
1963/67	0.0133	0.0163	0.0144	0.0114
1967/72	0.0695	0.0536	0.0536	0.0366
1972/77	0.0232	0.0230	0.0155	0.0249
Average	0.0325	0.0264	0.0243	0.0219

FINAL DEMAND COEFFICIENTS

	Demand	Supply	Horizontal	Vertical
1958/63	0.0116	0.0127	0.0072	0.0097
1963/67	0.0097	0.0081	0.0120	0.0131
1967/72	0.0362	0.1394	0.0534	0.1084
1972/77	0.1526	0.0708	0.0736	0.2562
Average	0.0525	0.0578	0.0366	0.0969

Table 2.9. Total Absolute Deviation as Percentage of Total Output (Case 2)

OUTPUT

	Demand	Supply	Horizontal	Vertical
1958/63	2.65	1.92	2.30	1.47
1963/67	2.27	1.61	1.57	1.42
1967/72	6.80	11.88	6.53	7.61
1972/77	5.29	3.60	4.13	5.36
Average	4.25	4.75	3.63	3.97

INTERINDUSTRY FLOW

	Demand	Supply	Horizontal	Vertical
1958/63	4.87	4.29	4.09	4.42
1963/67	3.59	3.07	3.11	3.01
1967/72	8.22	12.65	8.23	9.58
1972/77	6.02	4.78	5.35	6.09
Average	5.68	6.20	5.20	5.78

Table 2.10. Mean Percentage Deviation (Case 2)

OUTPUT

	Demand	Supply	Horizontal	Vertical
1958/63	6.00	2.47	2.26	2.10
1963/67	4.88	3.38	2.13	2.44
1967/72	7.53	10.09	6.28	4.11
1972/77	14.60	3.79	6.67	15.87
Average	8.25	4.93	4.34	6.13

the average total absolute interindustry flow deviation as percentage of total output, followed by the vertical, demand-side, and supply-side models. For all coefficient matrices and vectors, on average, the horizontal model performs better than the demand- and supply-side models. The vertical model, on the other hand, performs the best with the direct-input and value added coefficients, and the worst with the direct-output and final-demand coefficients. The results of Tables 2.8 - 2.10 clearly favor the horizontal model over the demand- and supply-side models, and on average, the former model outperforms the latter ones in all comparisons. The vertical model, on the other hand, shows a more erratic behavior with unclear conclusions.

How sensitive are the results in Tables 2.4 - 2.6 to the level of aggregation? Table 2.11 shows three indices of performance for the output vector at the 23-sector level.[1] The ranking of models according to their averages is similar to the previous results for output projections. The horizontal model performs better than the demand- and supply-side models for all projections and indices, while performing slightly better than the vertical model on average, confirming our previous results. We can conclude then, on the basis of the calculations in this section, that the generalized input-output model, and especially its horizontal version, tends to deliver superior forecasts for the U.S. economy in comparison to the existing input-output models.

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[1] The demand-supply mix parameters used for the two generalized models correspond to the ones used at the 7-sector level which are given in Table 2.3. The 7- and 23-sector classifications are listed in Appendix 1. Because of the redefinition of the scrap sector in the 1972 and 1977 tables, we have not included this sector in our calculations.

Table 2.11. Comparison of Output Projections at 23-sector level

MEAN ABSOLUTE DEVIATION

	Demand	Supply	Horizontal	Vertical
1947/58	3262	2888	1496	1988
1958/63	1373	1745	1203	1124
1963/67	1812	1912	1148	1476
1967/72	6776	11248	4951	3960
1972/77	9422	6830	6081	7007
Average	4529	4925	2976	3111

TOTAL ABSOLUTE DEVIATION AS PERCENTAGE OF TOTAL OUTPUT

	Demand	Supply	Horizontal	Vertical
1947/58	8.64	7.65	3.96	5.27
1958/63	2.84	3.61	2.49	2.32
1963/67	2.83	2.99	1.79	2.31
1967/72	7.09	11.77	5.18	4.14
1972/77	5.66	4.10	3.65	4.21
Average	5.41	6.02	3.41	3.65

MEAN PERCENTAGE DEVIATION

	Demand	Supply	Horizontal	Vertical
1947/58	12.25	8.79	5.97	7.58
1958/63	4.64	3.26	2.57	2.50
1963/67	4.11	3.58	2.32	3.13
1967/72	9.98	14.89	6.13	6.80
1972/77	11.11	6.18	4.95	6.38
Average	8.42	7.34	4.39	5.28

CHAPTER 3

INTRODUCING PRICES: THE PHYSICAL INPUT-OUTPUT TABLE

So far, our discussion of the generalized input-output model is based on the "nominal" table described by the identities (1) and (2). The coefficient matrices of the base year are derived from this "balanced" table, and the consistency condition we imposed ensures that these identities hold for the forecasted table. In this section, we discuss demand- and supply-side input-output models measured in physical units and with flexible prices. We then use this framework in the next section to formulate the physical generalized model. The "nominal" generalized model of previous sections can be conceived as a special case of the physical generalized model where the physical units for each sector of the base year are chosen such that the relative prices are a vector of ones, and are fixed to remain so over the projection through the consistency condition. The relationship between the nominal and physical variables (denoted by script notation) are:

$$Z = \hat{P}\mathcal{Z},$$

$$X = \hat{P}\mathcal{X},$$

$$Y = \hat{P}\mathcal{Y},$$

$$W = \mathcal{W},$$

where

$P = 1 \times n$ row vector of relative prices.

For reasons of simplicity, we assume a homogeneous value added vector that has the valuation 1 against which prices are expressed relatively. We also assume that, for an aggregated table, there exists an abstract "average price" for each sector. The relationship between the nominal and physical direct coefficient matrices and row coefficient vectors is shown below:

$$
\begin{aligned}
\mathcal{A} &= \mathcal{Z}\hat{\mathcal{X}}^{-1}, \\
&= (\hat{P}^{-1}Z)(\hat{P}^{-1}\hat{X})^{-1}, \\
&= \hat{P}^{-1}Z\hat{X}^{-1}\hat{P}, \\
&= \hat{P}^{-1}A\hat{P}, \\
\ell &= \mathcal{W}^{T}\hat{\mathcal{X}}^{-1}, \\
&= W^{T}(\hat{P}^{-1}\hat{X})^{-1}, \\
&= W^{T}\hat{X}^{-1}\hat{P}, \\
&= l\hat{P}, \\
\mathcal{B} &= \hat{\mathcal{X}}^{-1}\mathcal{Z}, \\
&= (\hat{P}^{-1}\hat{X})^{-1}(\hat{P}^{-1}Z) \\
&= \hat{X}^{-1}Z, \\
&= B, \\
d &= \mathcal{Y}^{T}\hat{\mathcal{X}}^{-1}, \\
&= (\hat{P}^{-1}Y)^{T}(\hat{P}^{-1}\hat{X})^{-1}, \\
&= Y^{T}\hat{X}^{-1}, \\
&= d.
\end{aligned}
$$

As Augustinovics (1970) has shown, the parameters of the physical supply-side model are independent of the price and valuation system. Although the supply-side model does not appear to have proper primal output and dual price systems as is apparent from Equation (41), the difference between the price systems of the demand- and supply-side models is not fundamental. We argue here that, over a projection, the two models are comparable in the sense that neither one can account for changes in output and price vectors at the same time without an additional equation.

The row identity of the physical input-output system is similar to identity (1):

$$(35) \qquad \mathcal{Z}i + \mathcal{Y} = \mathcal{X}.$$

We can no longer add the columns because each row is expressed in a different physical unit. There exists, however, a unique relative price vector that balances the input-output table vertically. We can rewrite identity (2) as:

$$(\hat{p}\mathcal{Z})^{T}i + \mathcal{W} = \hat{p}\mathcal{X},$$

$$\mathcal{Z}^{T}p^{T} + \mathcal{W} = \hat{\mathcal{X}}p^{T},$$

$$(36) \qquad p(\hat{\mathcal{X}} - \mathcal{Z}) = \mathcal{W}^{T},$$

The primal and dual equations of the physical demand-side input-output system are given by:

$$(37) \qquad \mathcal{A}\mathcal{X}^{D} + \mathcal{Y} = \mathcal{X}^{D},$$

$$(38a) \qquad p^{D}\mathcal{A} + \ell = p^{D},$$

$$(38b) \qquad p^{D}\mathcal{A} + \mathcal{W}^{T}\hat{\mathcal{X}}^{D-1} = p^{D},$$

and have the respective solutions:

$$(39) \qquad \mathcal{X}^{D} = (I - \mathcal{A})^{-1}\mathcal{Y},$$

$$(40) \qquad p^{D} = \mathcal{W}^{T}\hat{\mathcal{X}}^{D-1}(I - \mathcal{A})^{-1}.$$

The physical supply-side input-output model is given by:

$$(41) \qquad \mathcal{B}^{T}(\hat{p}^{S}\mathcal{X}^{S}) + \mathcal{W} = (\hat{p}^{S}\mathcal{X}^{S}),$$

which has the solution:

$$(42) \qquad (\hat{p}^{S}\mathcal{X}^{S}) = (I - \mathcal{B}^{T})^{-1}\mathcal{W}.$$

The "dual" of Equation (41), analogous to Equation (38), is:

$$(43a) \qquad \mathcal{B}i + d^{T} = i,$$

$$(43b) \qquad \mathcal{B}i + \hat{\mathcal{X}}^{-1}\mathcal{Y} = i.$$

The price system of the demand model, expressed by Equation (40), does not change over a forecast for any exogenously given $Y^* = \bar{Y}$, if both A and ℓ are kept constant by assumption. The implicit assumption of the demand-side model is that P^*, the forecasted relative price vector is fixed. However, under this assumption, through premultiplying Equation (42) by $\hat{P}^{-1}$ of the base year, the supply-side system can also solve for X^*. Hence, from this comparative static point of view, the difference between the two systems disappears. Over the forecast, Equation (38a) becomes a redundant equation like Equation (43a) of the supply-side model. However, expressed as Equations (38b) and (43b), they assure vertical consistency for the demand model and horizontal consistency for the supply model under fixed prices by assigning the "proper" endogenous W^* and Y^* vectors respectively.

All the input-output models analyzed in this study share the same taxonomy: two matrix equations that represent the row and column balances of the input-output table, given by Equations (35) and (36), and four unknowns (after the assumption about the behavior of the coefficient matrices are made): X, P, Y, and W. Assuming X is endogenous for all models, we can specify only one other variable endogenously. The nominal models of Chapter 2 assume that $P = i^T$, and besides the output equation, express the second equation by the consistency conditions (18), (19), and (26) for the demand-side, supply-side, and generalized models respectively. To construct the equivalent physical demand- and supply-side input-output models, we introduce prices explicitly, but assume (exogenous) constant prices over a forecast. Under fixed prices, either Y or W (or their convex combination in the case of the generalized model) must be calculated endogenously if the model is not to be overdetermined. It follows that, a model with endogenous prices requires that we specify both $\mathcal{Y}$ and $\mathcal{W}$ exogenously.[1] The demand-side and supply-side models are comparable under this specification as well. The solution to the demand-side model is clear from Equations (39) and (40). (Under exogenous $\mathcal{W}$, $\ell^* = \ell$ does not hold any longer.) The output solution for the supply model follows from Equation (43b):

$$X^S = \widehat{((I - \mathcal{B})i)}^{-1}\mathcal{Y}. \quad (44)$$[2]

Transposing and post-multiplying Equation (42) by $\hat{X}^{S-1}$, we get the solution for the relative prices for the supply-side model:

$$P^S = \mathcal{W}^T(I - \mathcal{B})^{-1}\hat{X}^{S-1}. \quad (45)$$

The "symmetry" between the two models is apparent. Output is determined by final demand, and given value added, the relative price vector is uniquely determined by that P^* that clears the markets by satisfying identity (36). Finally, this framework allows us to proceed with the physical generalized model. There are two ways to present this combination, both involving iterative solutions. The first one is less intuitive, but can be solved by an algorithm that exhibits global stability and converges relatively quickly. The second one shows the causalities at work more clearly but cannot be expressed as easily in an efficient algorithm. We present both in the next section.

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[1] Alternatively, ℓ^* can be derived explicitly from the projected technology matrix, and $\mathcal{W}^*$ given by $\hat{\ell}^* X^*$. This means adding a third equation to the system. The problem of endogenizing $\mathcal{W}$ according to a given production function is discussed in Chapter 4.

[2] Since $\mathcal{B}^* = \mathcal{B}$ for the supply-side model, it follows that $d^* = d$. Equation (44) is identical to: $X^S = \hat{d}^{-1}\mathcal{Y}$.

PHYSICAL GENERALIZED INPUT-OUTPUT MODEL

In order to take the convex combination of the output and price equations of the two models, we rewrite Equation (41) twice to generate two equations for output and price respectively:

$$(\hat{p}^{S-1}\mathcal{B}^T\hat{p}^S)X^S + \hat{p}^{S-1}\mathcal{W} = X^S, \tag{46}$$

$$p^S(\hat{X}^S\mathcal{B}\hat{X}^{S-1}) + \mathcal{W}^T\hat{X}^{S-1} = p^S. \tag{47}$$

The convex combination of Equations (37) and (46), under the constraint $X^D = X^S$, gives the primal equation of the generalized input-output model:

$$[\Lambda A^*X^* + (I-\Lambda)(\hat{p}^{*-1}\mathcal{B}^{*T}\hat{p}^*)X^*] + [\Lambda Y^* + (I-\Lambda)\hat{p}^{*-1}\mathcal{W}^*] = X^*. \tag{48}$$

For the horizontal version of the physical generalized model, we define the projected coefficient matrices A^* and $\mathcal{B}^*$ analogously to Equations (12) and (13). Taking the left-handed convex combination of the interindustry flow matrices:

$$Z^* = \Lambda Z^{D*} + (I-\Lambda)Z^{S*},$$

$$Z^* = \Lambda A\hat{X}^* + (I-\Lambda)\hat{X}^*\mathcal{B},$$

$$A^* = Z^*\hat{X}^{*-1},$$

$$A^* = \Lambda A + (I-\Lambda)\hat{X}^*\mathcal{B}\hat{X}^{*-1}, \tag{49}$$

$$\mathcal{B}^* = \hat{X}^{*-1}Z^*,$$

$$\mathcal{B}^* = (I-\Lambda)\mathcal{B} + \Lambda\hat{X}^{*-1}A\hat{X}^*. \tag{50}$$

Substituting Equations (49) and (50) into Equation (48) yields:

$$X^* = T^*X^* + S^*, \tag{51}$$

where

$$T^* = \Lambda^2 A + (I-\Lambda)\hat{p}^{*-1}\mathcal{B}^T\hat{p}^*(I-\Lambda) + (I-\Lambda)\hat{X}^*(\Lambda\mathcal{B} + \hat{p}^{*-1}A^T\hat{p}^*\Lambda)\hat{X}^{*-1},$$

$$S^* = \Lambda Y^* + (I-\Lambda)\hat{p}^{*-1}\mathcal{W}^*.$$

The algebraic manipulation, familiar by now, gives the reduced form:

$$X^* = T_r X^* + S^*, \tag{52}$$

where

$$T_r = \Lambda^2 A + (I-\Lambda)\hat{P}^{*-1}B^T\hat{P}^*(I-\Lambda) + (I-\Lambda)\widehat{[(\Lambda B + \hat{P}^{*-1}A^T\hat{P}^*\Lambda)i]}.$$

The primal output system solves for X^* as a function of P^*, Y^*, and W^*:

$$(53) \qquad X^* = (I - T_r)^{-1}S^*.$$

The next step is constructing the dual price system by taking the convex combination of Equations (38b) and (47) under the constraint $P^D = P^S$.

$$P^*A^*\Lambda + P^*(\hat{X}^*B^*\hat{X}^{*-1})(I-\Lambda) + W^{*T}\hat{X}^{*-1} = P^*,$$

$$P^*(I - A^*\Lambda - \hat{X}^*B^*\hat{X}^{*-1}(I-\Lambda)) = W^{*T}\hat{X}^{*-1},$$

$$(54) \qquad P^*(\hat{X}^* - A^*\hat{X}^*\Lambda - \hat{X}^*B^*(I-\Lambda)) = W^{*T}.$$

Substituting Equations (49) and (50), and solving:

$$(55) \qquad P^*(\hat{X}^* - \Lambda A\hat{X}^* - (I-\Lambda)\hat{X}^*B) = W^{*T}.$$

The solution to the dual price system of the horizontal generalized model is given as a function of X^*, and W^*:

$$(56) \qquad P^* = W^{*T}(\hat{X}^* - \Lambda A\hat{X}^* - (I-\Lambda)\hat{X}^*B)^{-1}.$$

For the vertical version, the projected coefficient matrices are derived from a right-hand side convex combination of Z^D and Z^S:

$$(57) \qquad A^* = A\Lambda + \hat{X}^*B\hat{X}^{*-1}(I-\Lambda),$$

$$(58) \qquad B^* = B(I-\Lambda) + \hat{X}^{*-1}A\hat{X}^*\Lambda.$$

Substituting Equations (57) and (58) into Equation (48), and solving for the reduced system, we obtain the primal output solution:

$$(59) \qquad X^* = (I - T_r)^{-1}S^*,$$

where

$$T_r = \Lambda A\Lambda + (I-\Lambda)^2\hat{P}^{*-1}B^T\hat{P}^* + \Lambda\widehat{[(B(I-\Lambda) + (I-\Lambda)\hat{P}^{*-1}A^T\hat{P}^*)i]}.$$

We derive the dual price equation of the vertical physical generalized model by substituting Equations (57) and (58) into Equation (54):

(60) $$P^* = \mathcal{W}^{*T}(\hat{X}^* - A\hat{X}^*\Lambda - \hat{X}^*\mathcal{B}(I-\Lambda))^{-1}.$$ [1]

Equations (53) and (56) constitute the general equilibrium system for the horizontal version, and Equations (59) and (60) make up the general equilibrium system for the vertical version of the physical generalized model. In both systems, the flexible price physical generalized input-output model is described by two simultaneous equations:

$$X^* = f(P^*, \bar{Y}, \bar{\mathcal{W}}),$$
$$P^* = f(X^*, \bar{\mathcal{W}}),$$

which are solved by iteration. Given $\Lambda = \bar{\Lambda}$, $Y^* = \bar{Y}$, and $\mathcal{W}^* = \bar{\mathcal{W}}$, we start the iteration with $P^* = P$ of the base year and iterate the pair of equations sequentially until the procedure converges to a "fixed point." The solution $\{X^*, P^*\}$ exists and is unique.[2]

The algorithm for solving the system of equations must deal explicitly with the case when any $\lambda_i = 0$. A zero sectoral demand-supply mix parameter means that the final demand for that sector will not be recorded anywhere in the system and the resulting forecast will not have balanced rows, that is identity (35) will not hold for that sector. When all $\lambda_i = 0$, the system can be solved by the supply-side model. When some $\lambda_i = 0$, the structure of the horizontal version implies that $d_i^* = d_i$, and x_i^* is given by $(\bar{y}_i/d_i)$. The structure of the vertical version, however, does not permit a similar treatment. For this case, after each iteration, we form the input-output table defined by the present value of X^* and P^*. (The

[1] Note that Equations (56) and (60) are both equivalent to $P^* = \ell^*(I - A^*)^{-1}$.

[2] An algorithm can be devised to shorten the number of iterations by adjusting the incremental change in P^* after each iteration according to the rate of convergence.

residually calculated final demand vector of these "transitional" tables are different than the exogenously given $\mathcal{Y}$ unless the iteration has already converged.) For all sectors with $\lambda_i = 0$, the information on final demand is "reinjected" after each iteration by augmenting the present value of x_i^* by the ratio $(\bar{y}_i / y_i^*)$ where y^* stands for the present "residual" value of final demand. In the final solution of the iteration process, if the residually calculated and exogenously given final demand and value added vectors are not identical, the algorithm must have arrived at a trivial solution with no economic meaning. We claim, without proving it here, that the physical generalized input-output system is stable with a unique solution if the base year technology is productive and the base year price and output vectors are nonnegative. (A possible method of proof is using some version of Brouwer's fixed point theorem.)

There is a fundamental reason why the same problem does not arise when some $\lambda_i = 1$. In fact, like the physical demand-side and supply-side models, the primal output of the physical generalized model is solely determined by final demand. An equivalent primal output system to Equation (48) is given by the convex combination of Equations (37) and (44):

$$(61) \qquad \mathcal{X}^* = \Lambda \mathcal{A}^* \mathcal{X}^* + [\Lambda + (I-\Lambda)(\widehat{(I - \mathcal{B}^*)i})^{-1}]\mathcal{Y}^*.$$

For the horizontal version, substitution yields:

$$(62) \quad \mathcal{X}^* = [I - \Lambda^2 \mathcal{A} - \widehat{\Lambda(I-\Lambda)\mathcal{B}i}]^{-1}[\Lambda + (I-\Lambda)(\widehat{(I- (I-\Lambda)\mathcal{B}-\Lambda\hat{\mathcal{X}}^{*-1}\mathcal{A}\hat{\mathcal{X}}^*)i})^{-1}]\mathcal{Y}^*.$$

For the vertical version, it becomes:

$$(63) \quad \mathcal{X}^* = [I - \Lambda\mathcal{A}\Lambda - \widehat{\Lambda\mathcal{B}(I-\Lambda)i}]^{-1}[\Lambda + (I-\Lambda)(\widehat{(I -\mathcal{B}(I-\Lambda)-\hat{\mathcal{X}}^{*-1}\mathcal{A}\hat{\mathcal{X}}^*\Lambda)i})^{-1}]\mathcal{Y}^*.$$

The new system of general equilibrium for the horizontal version is given by Equations (62) and (56), and for the vertical version by Equations (63) and (60), where

$$X^* = f(X^*, \bar{Y}),$$

$$P^* = f(X^*, \bar{W}).$$

The two equations are no longer simultaneous, and only the primal output equation is solved iteratively. However, the iteration poses problems when any diagonal entry of the second term on the right-hand side of Equations (62) and (63), which is a diagonal matrix, is driven to zero, generating meaningless solutions for that sector. The stability of the solution depends on the initial output vector with which the iteration starts. Although the unique solution of the previous system is still the unique meaningful solution to this system, the generation of this solution is not guaranteed under the latter. The stability of the former iterative system might be attributed to the feedback from prices although their effects drop out in the end. The latter system, however, shows the causalities at work accurately: output determined by final demand and prices determined by output and value added. Introducing an additional equation that relates final demand to value added and prices will render the system simultaneous, and allow the endogenous determination of final demand.[1]

In order to clarify the relationship between the nominal and physical generalized input-output models, we will explicitly derive here the consistency condition for the horizontal version. The row vector of coefficients for the physical generalized model is:

$$t^* = \ell^* \hat{P}^{*-1} \Lambda + d^* (I - \Lambda).$$

We get:

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[1] Our objective is to present the physical generalized input-output model in its basic form. The incorporation of the physical generalized input-output model in computable general equilibrium models, which we will not discuss here, constitutes an important extension of the present study.

$$\hat{t}^*X^* = \Lambda\hat{P}^{*-1}\hat{\ell}^*X + (I-\Lambda)\hat{d}^*X^*,$$

$$\hat{t}^*(I - T_r)^{-1}S^* = R^*,$$

where

$$R^* = \Lambda\hat{P}^{*-1}W^* + (I-\Lambda)Y^*.$$

After some algebra, we arrive at the following reduced form of the consistency condition:

$$R^* = KS^*, \tag{64}$$

where

$$K = [\widehat{[i^T(I - \Lambda\hat{P}^*A\hat{P}^{*-1}\Lambda - B^T(I-\Lambda)^2)]} - \Lambda[\hat{P}^{*-1}B^T\hat{P}^*(I-\Lambda) + (I-\Lambda)A]](I-T_r)^{-1}.$$

The relationship between W^* and Y^* is given by:

$$W^* = \hat{P}^*F_rY^*, \tag{65}$$

$$Y^* = F_r^{-1}\hat{P}^{*-1}W^*, \tag{66}$$

where

$$F_r = [\Lambda - K(I-\Lambda)]^{-1}[K\Lambda - (I-\Lambda)].$$

Under the nominal generalized model (where $P = P^* = i^T$), Equation (64) becomes identical to Equation (25). With fixed prices, the value added and final demand markets cannot be cleared for arbitrary values for both, and either Y^*, or W^*, or their convex combination (R^*) is given residually, as specified by Equations (65), (66), and (64), respectively.[1] Under flexible prices, on the other hand, Equation (64) holds for exogenously given Y and W through the fluctuation of prices and is implicitly implied in the equation system of the physical generalized model.

In this section, we have discussed the structure of the basic two-equation system of the flexible price generalized input-output model.

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[1] This holds for any fixed price projection where $P^* = \overline{P}$ (and not necessarily i^T).

However, the assumption that the projected generalized technology matrix, A^*, is compatible with any level of exogenously given value added violates the concept of production function. Later, we will consider three-equation systems where the value added vector, W^*, is determined endogenously according to a given production function, rather than residually (as in the nominal model) or exogenously (as in the two-equation system).[1]

PHYSICAL GENERALIZED INPUT-OUTPUT MODEL WITH NOMINAL EXOGENOUS VECTORS

When using the flexible price physical generalized model, there are occasions when we specify the exogenous final demand and value added vectors in nominal terms. An example is repeating the ex post forecasting of the U.S. input-output tables with the physical input-output models. The base year input-output table can be assumed to be a physical table with the base year price vector $P = i^T$. The latter year which is being forecasted, on the other hand, has a different set of prices by the assumption of our model, and its actual final demand and value added vectors are nominal quantities. If the final demand vector from the actual table is used with the primal equation of a physical input-output model, the forecast for the nominal final demand will differ from the actual one by a factor of sectoral relative prices. (Because the valuation of W is 1, no discrepancies occur between the exogenous and the forecasted nominal

[1] There is a basic asymmetry between the demand- and supply-side models: For $\Lambda = I$, $\ell^* = \ell$ need not hold, while for $\Lambda = 0$, $d^* = d$ must hold. The implication for the generalized horizontal model is that, it is not possible to drive it with value added and determine Y^* through a third equation, while the reverse can be done. This is because, once X^* is determined, Y^* is partly fixed for a given sector if $\lambda_i < 1$, and totally fixed if $\lambda_i = 0$. For the case where we drive the system with final demand, setting $\lambda_i = 0$ is equivalent to fixing the output level at $d_i y_i$.

value added vectors.) We can slightly change the model, however, and overcome this problem at the expense of losing the exact determination of the level of output and prices.

The primal output Equations (53) and (59) are expressed as:

$$(67) \qquad X^* = (I - T_r)^{-1}S^*.$$

where

$$S^* = \hat{P}^{*-1}(\Lambda Y^* + (I-\Lambda)W^*).$$ [1]

When coupled with the appropriate T_r and the dual equation (56) or (60) for the horizontal and vertical versions respectively, the respecification given in Equation (67) results in an infinite number of solutions for the model where each solution has the same relative output ratios and relative "relative price" ratios. Furthermore, the nominal solution ($\hat{P}^*X^*$) is constant for all solutions, and we can uniquely construct the nominal forecasted table for the purpose of comparing with the actual one. For any of the solutions, the "residually" calculated physical final demand multiplied by the price will give back the exogenously specified nominal final demand. In effect, this methodology permits us to fit the desired nominal final demand and value added levels to the forecast perfectly, and we will call it the fitted input-output model. Furthermore, by making an assumption about the "norm" of the forecasted relative price vector, based on some price versus wage index for example, we can solve the model for a unique forecasted physical input-output table as well.

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[1] This follows from identities $Y = \hat{P}\mathcal{Y}$ and $W = \mathcal{W}$. Note that, together they imply $S = \hat{P}\mathcal{S}$.

CHAPTER 4

GENERALIZED INPUT-OUTPUT MODEL AND THE PRODUCTION FUNCTION

The main criticism of the supply-driven model is that the resulting interindustry equilibrium may violate "the essential notion of production requirements, i.e., the production function." By association, the criticism extends to the generalized input-output model. We agree with Oosterhaven (1988) that the straightforward use of the supply-side model for impact studies of value added is problematic because of infeasible technological combinations that may arise. Indeed, one of the few applications of the generalized model under exogenously (i.e., arbitrarily) given value added that can be justified is the ex post forecasting we have undertaken in Chapter 2.

Earlier, we discussed the distinction made by Cronin (1984) between causal ordering and analytical assumption, and in Chapter 3, we formulated a two-equation system where the equilibrium level of output is determined by final demand only, and the equilibrium price vector is a function of value added.[1] In this section, we extend the physical horizontal generalized model to three equations where value added is determined endogenously as a function of the effective (i.e., quantity-constrained) technology matrix and the output level, according to a specified production function.

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[1] For $\Lambda = 0$, the generalized model falls back to the "degenerate" hybrid model of Cronin. This is indeed a most unrealistic case where all output levels are implicitly fixed: a perfectly planned economy where all inputs are determined by quantity rationing. However, as we have stressed earlier, the supply-side model is just a special case of the generalized model, and the main contribution of the generalized model is its continuous nature between the two extremes.

Going back to Equation (49), we can rewrite the horizontal technology matrix as:

$$A^* = \Lambda A + (I-\Lambda)GAG^{-1}, \tag{68}$$

where

$G = \hat{X}^*\hat{X}^{-1}$, the diagonal matrix of the growth factors of sectoral output.[1]

Under balanced growth, i.e., when all sectors have the same growth factor, the technology matrix remains unchanged. (This is also true for the allocation matrix.) The demand-side, supply-side, and generalized models are equivalent for this case. Under the more realistic assumption of unbalanced growth, the technology of sector i under the horizontal model is given by:[2]

$$\begin{bmatrix} \lambda_1 a_{1i} \\ \lambda_2 a_{2i} \\ \cdot \\ \cdot \\ \lambda_n a_{ni} \end{bmatrix} + \begin{bmatrix} (1-\lambda_1)(g_1/g_i)a_{1i} \\ (1-\lambda_2)(g_2/g_i)a_{2i} \\ \cdot \\ \cdot \\ (1-\lambda_n)(g_n/g_i)a_{ni} \end{bmatrix}$$

---

[1] Growth factor of a sector, g_i, is defined by 1 plus its growth rate.

[2] For the vertical model, the technology matrix $A^* = A\Lambda + GAG^{-1}(I-\Lambda)$, and the technology of sector i is given by:

$$\lambda_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \cdot \\ \cdot \\ a_{ni} \end{bmatrix} + (1-\lambda_i) \begin{bmatrix} (g_1/g_i)a_{1i} \\ (g_2/g_i)a_{2i} \\ \cdot \\ \cdot \\ (g_n/g_i)a_{ni} \end{bmatrix}$$

Assuming that the technology matrix, A, has not changed in the short run, the "projected" technology matrix given by Equation (68) can be interpreted as the effective constrained technology. Suppose that commodity j is supply-driven and all other commodities are demand-driven. If the output of sector j is growing faster than a given sector i, then the latter sector uses more of commodity j per unit of its output, something it could not do before because of the supply shortage. Because all of its other inputs are fixed, substitution takes place between value added and commodity j. For the case where sector j is growing relatively slower, substitution takes place in the other direction: more value added per unit of output is used. As λ_j moves away from 0 towards 1, the quantity constraint on sector i becomes "softer."

A link can be made here with the recent Keynesian macroeconomic models that emphasize product markets and nominal price rigidities.[1] Some causes of price rigidities include oligopolistic price setting, transaction costs of changing prices, imperfect information,[2] quasi-rational behavior, and price controls aimed at curbing inflation. Although the problem is usually studied in the framework of consumption, it can easily be applied to production. Under price rigidities, short run adjustments to clear markets will be made through quantities as well as prices. Here too, there is a continuum between completely flexible prices and completely fixed prices. Specifically, under upwardly rigid prices, excess demand will give rise to quantity rationing. The rule usually used here is that

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[1] See Benassy (1986), for example.

[2] It has also been argued that, simply the absence of a Walrasian auctioneer itself leads to a non-Walrasian equilibrium.

all agents are rationed proportionally once they express their notional demands.[1] In contrast, in the horizontal generalized model, we assume that a sector's effective demand for a given intermediate input is a convex combination of the sector's current notional demand and its past market share of that intermediate good. The connection between the horizontal generalized input-output model and the fixed price equilibrium approach remains to be investigated. A possibility is to fix the prices of the supply-driven sectors in the horizontal generalized model and determine the corresponding demand-supply mix parameters endogenously.

Below, we will consider three specifications for a production function. We assume that the sectors, faced with their effective intermediate input vector, A^*_j, choose that value of ℓ^*_j, which satisfies the constraint implied by a production function.[2] The system is given by:

$$\mathcal{X}^* = f(X^*, \bar{\mathcal{Y}}),$$

$$P^* = f(\mathcal{X}^*, \mathcal{W}^*),$$

$$\mathcal{W}^* = f(\mathcal{P}, A^*, \mathcal{X}^*),$$

where $\mathcal{P}$ is the set of the production functions of n sectors.

The first alternative we will consider is a two-level production function. At the first stage, sectors express their notional demands for intermediate inputs according to a Leontief function (given by their base year technology, A_j). At the second stage, their effective demand given by the column j of the quantity-constrained technology matrix, A^*,

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[1] Notional demand of an agent is that which s/he would express in the absence of any quantity rationing. Effective demand is that which the agent expresses when s/he perceives these quantity constraints.

[2] Clearly, we cannot consider the Leontief function here, because it does not allow substitution between inputs.

determines the unit value added input, ℓ_j^*, according to a Cobb-Douglas function between all inputs. The unit production function for sector j is expressed by:

$$(69) \qquad P_j(1) = H_j(\Pi_{i=1}^{n} A_{ij}{}^{a_{ij}})\ell_j{}^{l_j},$$

where a_{ij} and l_j are the value share parameters (same as the base year value input-output coefficients) and H_j is the shift parameter. Hence, ℓ_j^* is determined according to Equation (69):

$$\ln 1 = \ln H_j + \Sigma_{i=1}^{n} a_{ij} \ln \alpha_{ij} + l_j \ln \ell_j,$$

$$\ln 1 = \ln H_j + \Sigma_{i=1}^{n} a_{ij} \ln \alpha_{ij}^* + l_j \ln \ell_j^*.$$

Subtracting the second from the first, and solving for ℓ_j^*, we obtain:

$$(70) \qquad \ell_j^* = \ell_j \, e^{-(1/l_j)\Sigma_{i=1}^{n} a_{ij} \ln(\alpha_{ij}^*/\alpha_{ij})} \qquad (j=1,\ldots,n).$$

This formulation has the drawback that we assume without justification two different production functions. The implicit assumption is a Cobb-Douglas production function between value added and a composite supply-driven intermediate input and a Leontief production function for a composite demand-driven input. Hence, the implied production function depends on Λ. We can overcome this problem by assuming a Cobb-Douglas function between all inputs.

De Boer (1976) has shown that the constant value proportions of inputs assumed by the nominal (demand-side) input-output model can only be derived if the underlying production function is of the linear-homogenous Cobb-Douglas type. For the physical (demand-side) input-output model, the above is equivalent to fixing the physical projected technology matrix , A^*, at $\hat{P}^{*-1}A\hat{P}^*$, rather than at A. Implanting this to the equation system of the horizontal model, we get:

$$(71) \qquad \mathcal{X}^* = [I - \Lambda^2\hat{P}^{*-1}A\hat{P}^* - \widehat{\Lambda(I-\Lambda)\mathcal{B}i}]^{-1} \times$$

$$[\Lambda + (I-\Lambda)(\widehat{(I- (I-\Lambda)\mathcal{B} - \Lambda\hat{\mathcal{X}}^{*-1}\hat{P}^{*-1}A\hat{P}^*\hat{\mathcal{X}}^*)i})^{-1}]\mathcal{Y}^*,$$

$$(72) \qquad P^* = \mathcal{W}^{*T}(\hat{\mathcal{X}}^* - \Lambda\hat{P}^{*-1}A\hat{P}^*\hat{\mathcal{X}}^* - (I-\Lambda)\hat{\mathcal{X}}^*\mathcal{B})^{-1}.$$

The system is fully interdependent:

$$\mathcal{X}^* = f(X^*, P^*, \bar{\mathcal{Y}}),$$

$$P^* = f(\mathcal{X}^*, \mathcal{W}^*),$$

$$\mathcal{W}^* = f(\mathcal{P}, A^*, P^*, \mathcal{X}^*),$$

where $\mathcal{P}_j(1)$ is given by (69) and ℓ_j^* is determined for each sector according to Equation (70). The implicit assumption here is that the unfulfilled demand arising from quantity rationing "spills over" to value added.

A third, and more appropriate, variant can be based on the generalized Leontief function that was first introduced by Diewert (1971). The generalized Leontief function has the desirable property of being able to attain any set of partial elasticities of substitution among inputs. We can express the generalized Leontief function in our context as follows:

$$(73) \qquad \mathcal{P}_j(1) = 1/\rho(\hat{E}_j^{-1/2}C_j\hat{E}_j^{-1/2}), \qquad (j=1,\ldots,n),$$

where,

E_j is the n+1 unit input vector of sector j, containing the unit value added, ℓ_j, in its n+1st row,

C_j is a n+1×n+1 symmetric matrix which contains the parameters of the production function of sector j,

ρ is the spectral radius.

The partial elasticity of substitution between inputs i and j is given by:

$$\sigma_{ij} = \frac{c(P)(\partial^2 c(P)/\partial p_i \partial p_j)}{(\partial c(P)/\partial p_i)(\partial c(P)/\partial p_j)}$$

where c(P) is the generalized Leontief cost function associated with (73):

$$\Sigma_{i=1}^{n}\Sigma_{j=1}^{n}c_{ij}p_i^{1/2}p_j^{1/2}.$$

Hence, the larger the parameter c_{ij}, the larger is the partial elasticity of substitution between inputs i and j for a given sector ($c_{ij} = 0$ implies $\sigma_{ij} = 0$). When all off-diagonal parameters are zero, the generalized Leontief function is equivalent to the ordinary Leontief function. For our purposes, we will assume positive partial elasticities of substitution between intermediate goods and value added only.[1]

For the generalized horizontal model given by Equations (62) and (56), we add the following equation:

$$(74) \qquad W=\hat{\ell}^{*}X^{*},$$

with ℓ_j^* subject to:

$$\rho(\hat{E}_j^{*-1/2}C_j\hat{E}_j^{*-1/2}) = 1, \qquad (j=1,\ldots,n),$$

where $E_j^* = \begin{bmatrix} A_j^* \\ \ell_j^* \end{bmatrix}$

In the first stage the demand for intermediate goods take place according to a Leontief function, and in the second stage the unfulfilled intermediate demands are substituted with value added according to the production function constraint given in (74). Because the output level is

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[1] The matrix C_j of sector j takes the form:

$$\begin{bmatrix} c_{11} & & & & c_{1v} \\ & c_{22} & & & c_{2v} \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & c_{nn} & c_{nv} \\ c_{v1} & c_{v2} & \cdot & c_{vn} & c_{vv} \end{bmatrix}$$

where v stands for value added and $c_{iv} = c_{vi}$.

not affected by prices, no new quantity constraints are introduced and equilibrium prevails.

In Table 4.12, we compare the projections of the horizontal generalized model under the three functional forms we have introduced here. The models are based on the 1963 U.S. coefficient matrices and vectors along with the 1967 final demand vector.[1] The percentage change in output and value added coefficients are expressed against the base year values. The new relative prices are also listed. For each model, we show the impacts for several specifications of Λ.[2] The columns represent the sectors of the 7-sector classification. LCD, CD, and GLF stand respectively for two-level Leontief/Cobb-Douglas, Cobb-Douglas, and generalized Leontief functions.

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[1] Unlike the Cobb-Douglas function, the parameters of the generalized Leontief function cannot be point estimated. We have assigned the following parameters:

$$C_j = \begin{bmatrix} .5a_{1j} & & & & & .5a_{1j} \\ & .5a_{2j} & & & & .5a_{2j} \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & & & & .5a_{nj} & .5a_{nj} \\ .5a_{1j} & .5a_{2j} & \cdot & \cdot & .5a_{nj} & \ell_j - .5\Sigma_{i=1} a_{ij} \end{bmatrix} \qquad (j=1,\ldots,n).$$

This quite arbitrary calibration results in partial elasticities of substitution between .4 and 2.6. In general, parameters of a production function should be carefully calibrated by taking the effect of quantity constraints into account.

[2] Note that when the demand-suply mix parameters of all sectors are identical, the vertical and horizontal models become equivalent. $\bar{\Lambda}$ is a diagonal matrix of the demand-supply mix parameters for the horizontal model listed in Table 2.3.

Table 4.12. Impact of the 1967 Final Demand Using the 1963 Table: Forecasts of 1967 Output, Price, and Value Added Coefficients

Forecasted Price (LCD)

	1	2	3	4	5	6	7
$\Lambda =$ I	1.	1.	1.	1.	1.	1.	1.
$\Lambda =$ .67*I	1.0002	1.0001	1.0007	1.0001	1.0000	1.0001	1.0002
$\Lambda =$ .33*I	1.0023	1.0003	1.0033	1.0006	1.0002	1.0004	1.0010
$\Lambda =$ .15*I	1.0076	1.0006	1.0063	1.0018	1.0004	1.0010	1.0021
$\Lambda =$ 0*I	1.0264	1.0039	1.0115	1.0062	1.0012	1.0024	1.0054
$\Lambda = \bar{\Lambda}$	1.0061	1.0011	1.0029	1.0041	1.0006	1.0012	1.0027

Forecasted Price (CD)

	1	2	3	4	5	6	7
$\Lambda =$ I	1.	1.	1.	1.	1.	1.	1.
$\Lambda =$ .67*I	1.0002	1.0001	1.0007	1.0001	1.0000	1.0001	1.0002
$\Lambda =$ .33*I	1.0024	1.0003	1.0034	1.0006	1.0002	1.0004	1.0010
$\Lambda =$ .15*I	1.0071	1.0006	1.0065	1.0019	1.0005	1.0010	1.0022
$\Lambda =$ 0*I	1.0264	1.0039	1.0115	1.0062	1.0012	1.0024	1.0054
$\Lambda = \bar{\Lambda}$	1.0062	1.0011	1.0029	1.0041	1.0006	1.0012	1.0027

Forecasted Price (GLF)

	1	2	3	4	5	6	7
$\Lambda =$ I	1.	1.	1.	1.	1.	1.	1.
$\Lambda =$ .67*I	1.0006	1.0001	1.0011	1.0002	1.0001	1.0002	1.0004
$\Lambda =$ .33*I	1.0053	1.0008	1.0060	1.0019	1.0006	1.0013	1.0027
$\Lambda =$ .15*I	1.0166	1.0022	1.0119	1.0064	1.0015	1.0031	1.0068
$\Lambda =$ 0*I	1.0547	1.0133	1.0263	1.0264	1.0046	1.0015	1.0206
$\Lambda = \bar{\Lambda}$	1.0170	1.0046	1.0114	1.0210	1.0031	1.0061	1.0139

% Change in Value Added Coefficients (LCD)

	1	2	3	4	5	6	7
$\Lambda = I$	0.	0.	0.	0.	0.	0.	0.
$\Lambda = .67*I$	-1.62	0.07	-3.93	0.22	0.06	0.29	0.80
$\Lambda = .33*I$	-5.06	0.39	-8.51	0.83	0.16	0.66	1.96
$\Lambda = .15*I$	-9.04	0.48	-11.20	1.75	0.25	0.94	2.54
$\Lambda = 0*I$	-16.18	5.35	-13.71	2.85	0.34	1.13	3.90
$\Lambda = \bar{\Lambda}$	-7.10	2.04	-4.46	2.75	0.00	0.36	2.01

% Change in Value Added Coefficients (CD)

	1	2	3	4	5	6	7
$\Lambda = I$	0.	0.	0.	0.	0.	0.	0.
$\Lambda = .67*I$	-1.63	0.07	-3.98	0.22	0.07	0.29	0.81
$\Lambda = .33*I$	-5.14	0.41	-8.64	0.85	0.17	0.67	1.94
$\Lambda = .15*I$	-9.22	0.52	-11.30	1.79	0.26	0.94	2.53
$\Lambda = 0*I$	-16.18	5.35	-13.71	2.85	0.34	1.13	3.90
$\Lambda = \bar{\Lambda}$	-7.23	2.06	-4.48	2.67	0.04	0.39	1.94

% Change in Value Added Coefficients (GLF)

	1	2	3	4	5	6	7
$\Lambda = I$	0.	0.	0.	0.	0.	0.	0.
$\Lambda = .67*I$	-1.57	0.07	-3.83	0.23	0.07	0.30	0.84
$\Lambda = .33*I$	-4.64	0.42	-8.06	0.97	0.18	0.72	2.17
$\Lambda = .15*I$	-7.79	1.18	-10.42	2.16	0.28	1.06	3.41
$\Lambda = 0*I$	-12.58	6.29	-12.49	5.37	0.42	1.58	5.71
$\Lambda = \bar{\Lambda}$	-6.06	2.18	-4.16	5.07	0.04	0.66	3.40

	% Change in Output (LCD and GLF)						
	1	2	3	4	5	6	7
$\Lambda =$ I	27.48	34.72	23.88	34.08	35.41	35.81	35.38
$\Lambda =$ .67*I	25.41	35.36	23.22	34.43	35.81	36.12	35.69
$\Lambda =$ .33*I	20.67	36.72	22.41	34.91	36.32	36.48	35.74
$\Lambda =$ .15*I	14.93	38.83	21.92	35.25	36.65	36.70	35.92
$\Lambda =$ 0*I	4.40	50.36	21.49	35.63	37.00	36.92	36.13
$\Lambda = \bar{\Lambda}$	4.40	50.36	23.84	34.43	35.15	36.13	35.56

	% Change in Output (CD)						
	1	2	3	4	5	6	7
$\Lambda =$ I	27.48	34.72	23.88	34.08	35.41	35.81	35.38
$\Lambda =$ .67*I	25.40	35.37	23.21	34.43	35.81	36.12	35.68
$\Lambda =$ .33*I	20.55	36.78	22.39	34.92	36.33	36.49	35.79
$\Lambda =$ .15*I	14.66	40.00	21.90	35.29	36.67	36.72	35.99
$\Lambda =$ 0*I	4.40	50.36	21.49	35.63	37.00	36.92	36.13
$\Lambda = \bar{\Lambda}$	4.40	50.36	23.82	34.40	35.26	36.17	35.48

For demand-side technology, all models are equivalent. Under our input-output framework, prices are determined by costs (as a function of the effective technology matrix), and relative prices stay the same for all combinations of final demand proportions for $\Lambda = I$. (This is the "classical" notion of prices of production.) For $\Lambda < I$, on the other hand, technology is partly a function of output proportions. Hence, as the physical input-output model becomes more supply-driven, shifts in the composition of final demand affect relative prices to a greater extent through shifts in output proportions. Furthermore, for fixed final demand, $\partial P^*/\partial \lambda_i$ is larger in absolute value for smaller λ_i. There is a nonlinear increase in the "distortions" as the demand-supply mix parameters decrease in value. In fact, an index of the forecasted price vector is a good measure of the adverse effect of quantity constraints. The rise in the price level is due to employing more value added in order to produce the same bundle of final demand. In the presence of unbalanced growth, similar impacts give rise to greater disturbances for relatively supply-driven systems where the equilibrium states are relatively more unstable. (We will further investigate the relation between technology and unbalanced growth in the next two sections.)

TECHNOLOGY AND GROWTH

In this section, we examine the interdependence between technology and growth in the generalized physical input-output model and suggest some interpretations of the biproportionality relation between the demand- and supply-side technology matrices. We then use the framework in the next section to construct a simple generalized dynamic growth model.

For the long run, a plausible explanation exists for nondemand-side technologies. Given the demand-supply mix parameters, the technology is a continuous function of relative growth factors. For $\Lambda < I$, the higher the relative growth factor of a sector, the less it needs the inputs from slower-growing sectors to produce a unit of its commodity. The causality here is working backwards: technological progress in a sector (in the sense of requiring less inputs for the same amount produced) results in a higher relative growth factor for that sector.[1] This is also true for the vertical technology.

If we denote by Γ the set of all feasible growth factors, then the corresponding technology for a given sector becomes a closed convex cone in the nonnegative orthant of R^n (the n-dimensional Eucledian space).[2] The center of the cone is given by A_i and the boundary points of the cone are given by GA_ig_i. The central ray represents demand-side (as well as balanced growth) technology. As λ_i goes from 1 to 0, the feasible technology region expands. For the supply-side technology, it consists of the whole cone, including the boundary points. Because both horizontal and vertical technologies are bounded by GA_ig_i, they span the same feasible technology space. For a particular Λ, however, the associated cone will be different for the two models. If both Λ and $G \epsilon \Gamma$ are given, then the associated technology becomes a unique ray of the cone. In other words, for each growth path, the technology is fixed for given Λ.

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[1] This, of course, does not hold under the demand-side model where the technology is kept unchanged by assumption. We will later show that, for $\Lambda < I$, the higher the relative growth rate of a sector, the lower is its relative price. Again, this reflects technological progress in the form of cost-efficiency.

[2] The conical property implies constant returns to scale, convexity implies that the weighted average of two feasible technologies is also feasible, and the fact that the set is closed implies that the boundary points are included in the set.

A DYNAMIC FORMULATION

In order to investigate the relationship between technology and growth in the generalized input-output model further, we will first present a demand-side dynamic growth model and its conventional balanced growth solution. This von Neumann-Leontief type model is then extended to our framework under generalized technology and unbalanced growth path.

Von Neumann (1945)[1], in his analysis of an expanding multisector economy, proved the existence of a maximum attainable rate of steady growth of the economy where all sectors grow at the same maximal rate, all prices and the interest rate remain constant, and the maximal balanced growth rate equals the interest rate. The model is closed in the sense that labor is regarded not as a primary resource but as produced by the system like any other commodity where consumption is the input and labor is the output of the household sector. There is no outside consumption in the system and the producers reinvest all surplus. There is no fixed capital: all commodities are produced by commodities. There exists perfect competition and rates of return in each sector equals the rate of interest. The original model allows joint production and more than one activity to produce a commodity. A simplified von Neumann-Leontief model is presented here with no joint production, one activity for a sector, and an irreducible technology matrix.[2]

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[1] The original paper, in German, was published in 1937. For a detailed presentation of the von Neumann growth model and its extensions, see Nikaido (1968), Chapter 2.

[2] Irreducibility means that no process in the model can sustain production unless every commodity in the economy is produced. There exists no proper subset of commodities that can be produced using inputs exclusively from

Let us suppose that the production of each commodity takes one period. At the beginning of period t+1 (i.e., at the end of period t), the amount of good i that is available is $X_i(t)$. This is used for production in period t+1. The following discrete-time difference equation gives the primal equation:

$$AX(t+1) = X(t). \tag{75}$$

This is the technological assumption that states that the input requirements for period t+1 equal the available amounts from the preceding period. The dual equation is given by:

$$\beta(t)p(t)A = p(t+1), \tag{76}$$

where

β = interest factor, i.e., 1 plus the interest rate.

p = n $\times$1 relative price vector.[1]

The producers of commodity i, in order to produce one unit of that commodity, must purchase quantities a_{ji} of commodities j, j=1,...,n, each at unit price $p_i(t)$. At the end of the production period, they sell one unit of i at price $p_i(t+1)$. The rate of return, β-1, is assumed to be uniform across the sectors and alternatively, under competitive assumptions, can be interpreted as the interest rate. Equation (76) is the economic assumption of the von Neumann-Leontief model, and states that each production process is financed by borrowing and yields zero economic profit.

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the same subset, that is, without using at least one input that is not in the subset.

[1] Unlike the price vector P of the earlier sections, this relative price vector is not relative to a given valuation of value added and can be scaled to any size.

Is the homogeneous system (75)-(76) capable of balanced growth by a growth factor γ, such that $X(t+1) = \gamma X(t)$, $p(t+1) = p(t)$, and $\beta(t) = \beta$ (constant)? Rewriting the system:

$$\gamma A x = x, \tag{77a}$$

$$(\xi I - A)x = 0, \tag{77b}$$

$$\beta p A = p, \tag{78a}$$

$$p(\xi I - A) = 0, \tag{78b}$$

where

x = the relative output vector,

$\xi = 1/\gamma = 1/\beta$, the reciprocal of the growth and interest factors.

The set of equations (77b) and (78b) has a nontrivial solution if and only if the determinant:

$$\det(\xi I - A) = 0. \tag{79}$$

This is a polynomial in ξ and has n roots, which are the eigenvalues of A. Each solution is associated with a distinct right eigenvector (output proportions), and a left eigenvector (price proportions).[1] A well-known property of an irreducible nonnegative matrix A is that its dominant eigenvalue, $\rho(A)$, is real, positive, largest in absolute value, and is the only eigenvalue associated with nonnegative left and right eigenvectors. Hence $\xi = \rho(A)$ is the only solution that is economically meaningful. This proves that a unique exponential balanced growth solution exists where

[1] For $\det(\xi I - A)$ to be zero, its rank must be less than n, i.e., the matrix must be singular. The output solution, x, lies in its nullspace, and the price solution, p, lies in its left nullspace. An eigenvector is determined only up to a scalar multiplier. We will assume both x and p are scaled such that their sum equal one, i.e., they lie on the unit simplex. (In this section, we are strictly interested in relative output and price vectors.) Note that $x^* = Gx/\Sigma(Gx)$.

$\gamma = \beta = 1/\rho(A) > 0$, $x > 0$, $p > 0$. The solution set $\{\gamma, \beta, x, p\}$ is called the balanced growth von Neumann-Leontief quadruplet. If $\rho(A) < 1$, then A is productive: the economy is capable of expanding at a positive growth rate of $\gamma-1$; if $\rho(A) > 1$, then the economy decays; and if $\rho(A) = 1$, it is just capable of reproducing itself.[1]

Equation (75) is a backward difference equation. Hence, it is the eigenvalues of A^{-1} rather than A that govern the expansion. (The economic interpretation of this is that the von Neumann-Leontief growth factor γ is the growth factor of the slowest growing sector.) The balanced growth factor γ is the inverse of the largest eigenvalue of A. Because the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, the corresponding eigenvalue of A^{-1}, γ, is the smallest (in absolute value) of all eigenvalues of A^{-1}. This means that, if the initial conditions do not exactly correspond to the balanced von Neumann-Leontief quadruplet, the balanced growth path, in time, will be dominated by one of the other solutions. (Under irreducibility, all the other solutions are negative.) Therefore, the balanced growth path of the von Neumann-Leontief model is locally unstable: small deviations from the balanced path are amplified. Actual growth based on the assumptions of the model will be unbalanced, and it is only logical that the model should be extended to incorporate unbalanced growth.

[1] Note that the growth factor γ is that value that will render $(I - \gamma A)$ a singular M-matrix as $\rho(\gamma A) = \gamma\rho(A) = 1$. Unlike the open Leontief model, there is no outside consumption, and singularity implies the full utilization of produced goods. If $\rho(\gamma A) > 1$, $(I - \gamma A)$ is no longer an M-matrix, implying the infeasibility of that growth factor.

The generalized unbalanced growth von Neumann-Leontief model presented below is based on the technological assumptions of the generalized input-output model. Clearly, a linear unbalanced growth model cannot maintain fixed proportions for its sectoral outputs and is inherently unstable. In that regard, however, such a model is not so different from the balanced growth von Neumann-Leontief model, because the latter cannot maintain itself either; on the other hand, the generalized version has two main advantages. First, it identifies the feasible (and efficient) growth path possibilities of the economy. Second, it introduces nonlinearities to the von Neumann-Leontief model and under carefully chosen specifications may lead to limit cycles.

We start with the same set of difference equations (75) and (76), and investigate the possible unbalanced growth paths under the horizontal technology given from Equation (68), such that $X(t+1) = GX(t)$, $p(t+1) = p(t)$, and $\beta(t) = \beta$ (constant):[1]

(80) $$A^{*}Gx = x,$$

(81) $$\beta p A^{*} = p.$$

Together, Equations (80) and (81) result in:

$$pGx = \beta p x,$$

---

[1] This is the simplest unbalanced model we can construct. Alternatively, different rates of return can be assumed for each sector. The growth factor in one period, then, can be related to the return factor of the preceding period, for each sector. Another possibility is to distribute the surplus to sectors as some decreasing function of their demand-supply mix parameters, reflecting the degree of oligopoly in that sector for example. At any rate, the main objective of this section is to suggest how the generalized model can be formulated in a dynamic framework, rather than actually constructing a model with a unique solution.

the value of output at the end of the production period equals the value of output at the beginning of the period plus the rate of return on it. Substituting A^* from Equation (68) and rearranging, the horizontal von Neumann-Leontief system is given by:

(82) $$(I - (\Lambda A G + (I-\Lambda) G A))x = 0,$$

(83) $$p(I - (\Lambda A + \beta (I-\Lambda) G A G^{-1})) = 0.$$

The model has a nontrivial solution only when Equations (84) and (85) hold for its primal and dual equations respectively:

(84) $$\det(I - (\Lambda A G + (I-\Lambda) G A)) = 0,$$

(85) $$\det(I - \beta(\Lambda A + (I-\Lambda) G A G^{-1})) = 0.$$ [1]

These two equations are equivalent to $\rho(\Lambda A G + (I-\Lambda) G A) = 1$, and $\rho(\beta(\Lambda A + (I-\Lambda) G A G^{-1})) = 1$, respectively. The former generates the feasible and efficient growth factor vectors $G \in \Gamma$, and the latter pairs each growth factor vector with an interest factor. Clearly, $G = \gamma I$, the balanced growth factor, is a solution to Equation (84). If the growth factor of one of the sectors is increased while the others are held at γ, the determinant in Equation (84) will be negative. This signifies an infeasible growth path, and at least one of the remaining sectors' growth factors must be lowered such that Equation (84) holds. (For a productive economy, it is convenient to consider only $G \geq I$.) The set of all diagonal matrices $G \geq I$ which satisfy Equation (84) constitute the growth frontier, Γ. Under irreducibility, this is a continuous convex bounded hypersurface in R^n,

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[1] For the vertical model, Equations (82) and (83) become:

$$(I - (A G \Lambda + G A (I-\Lambda)))x = 0,$$

$$p(I - \beta(A\Lambda + G A G^{-1}(I-\Lambda))) = 0.$$

centered at γI, and delineating the infeasible growth region from the inefficient growth region. At all points on the frontier, it is not possible to increase any one sector's growth factor without decreasing the growth factor of at least one other.

In order to clarify the properties of the generalized growth model, we first consider the special cases of demand-side and supply-side technologies.

For $\Lambda = I$, the Equations (84) and (85) become:

(86) $$\det(I - \hat{A}G) = 0,$$

(87) $$\det(I - \beta\hat{A}) = 0.$$

The output solution, x, is given by the eigenvector associated with $\rho(\hat{A}G) = 1$, and varies with each G. The dual equation (87) is identical with the balanced von Neumann-Leontief model and has the same solution $\{\beta, p\}$.

For the case $\Lambda = 0$, $\hat{A}^* = G\hat{A}G^{-1}$, and the supply-side equations are given by:

(88) $$\det(I - G\hat{A}) = 0,$$

(89) $$\det(I - \beta G\hat{A}G^{-1}) = 0.$$

Equation (88) gives rise to the same growth frontier as the demand-side model because both primal equations are equivalent to:

$$\det(G^{-1} - \hat{A}) = 0.$$

The associated eigenvector, x, however, is different for each case. For all $G \in \Gamma$, Equation (89) has the same solution as the demand-side model ($\beta = 1/\rho(\hat{A})$), because $\hat{A}$ and $G\hat{A}G^{-1}$ are similar matrices and therefore share the same eigenvalues. Although β is invariant with G, unlike the demand-side model, the solution to p depends on the particular growth factor vector G.

The sets of (infinitely many) solutions $\{G, \beta, x, p\}$ that satisfy Equations (82) and (83) are called the generalized von Neumann-Leontief quadruplets. For a given Λ and a growth factor vector G that satisfies Equation (84), however, the solution is a unique quadruplet. The growth frontier for a given Λ between 0 and I is different than the demand-side and supply-side growth frontiers. The solution β to Equation (85) is also irregular: it is no longer constant and varies with G. The growth factor frontier can be visualized as a frisbee spinning at its center, the one fixed point that never changes (corresponding to $G = \gamma I$, the balanced growth solution). Although $\Lambda = 0$, and $\Lambda = I$ result in the same position, for any Λ in between, the surface moves as a continuous function of Λ. Each point on the surface is associated with a unique quadruplet.

The generalized von Neumann-Leontief model, has important implications for planning: it computes alternative growth paths available under a given Λ, along with the output proportions, price proportions, and the interest rate that support each path. The model can also be used in establishing comparative statics relationships among growth, output and price proportions.

There are well-defined relationships between the demand- and supply-side models, due to the fact that they share the same growth frontier. The output solution for the demand-side model, x^D, is the eigenvector of $\hat{A}G$ corresponding to the dominant root, which is 1 for $G \in \Gamma$. Premultiplying Equation (80) for the demand-side model by G, we obtain $(G\hat{A})Gx^D = Gx^D$. This, however, is the primal equation for the supply-side model and shows that, for fixed G, x^S is proportional to Gx^D. For the same growth path, the relative output proportions of the higher growing sectors are larger for the supply model. Conversely, for more unbalanced growth paths, the differences in output proportions for the two models are greater.

The price vector for the demand model is given by the eigenvector of A^T associated with its dominant eigenvalue, $1/\beta$. For the supply model, the price vector is the eigenvector of $G^{-1}A^TG$, associated with the same eigenvalue.[1] Let's write A^T in its diagonal form: $A^T = HJH^{-1}$, where H has the n eigenvectors on its columns, and J has the corresponding eigenvalues on its diagonal entries. For the supply-side model, we obtain:

$$G^{-1}A^TG = G^{-1}(HJH^{-1})G = (G^{-1}H)J(G^{-1}H)^{-1}.$$

The matrix $G^{-1}H$, then, carries the eigenvectors of $G^{-1}A^TG$. Hence, for a given G, p^S is proportional to $G^{-1}p^D$. Furthermore, because the demand-side technology is fixed, p^D is constant and is invariant over the growth factor frontier. Therefore, for each quadruplet of the supply model, Gp^S is proportional to p^D. Along the growth frontier of the supply model, as relative growth of a sector increases with respect to another sector, their price ratio decreases. Finally, taken together, these relations imply that x^D is proportional to $\hat{p}^S x^S$ for a given $G \in \Gamma$. For the demand-side model, the effect of unbalanced growth is completely absorbed by output, while for the supply-side model both output and prices will adjust.

Generalizing these relations to $0 < \Lambda < I$, for a given growth factor, as the model becomes more supply-driven, the relative output proportions of faster-growing sectors will be higher and their relative prices will be

[1] The eigenvalues of a matrix and its transpose are identical and the right eigenvectors of one are identical with the left eigenvectors of the other.

lower. For fixed $\Lambda < I$, faster growth implies lower price in relative terms. Appendix 3 lists various solutions for all versions of the unbalanced growth von Neumann-Leontief model, using hypothetical data for a 3-sector economy.

Obviously, the generalized von Neumann-Leontief model presented in this section is too simplistic to be used in actual planning. However, it is the basic building block of a generalized open dynamic Leontief input-output model with fixed capital and consumption. Furthermore, the introduction of an additional difference equation relating sectoral demand-supply mix parameters to growth rates and nonuniform sectoral rates of return might lead to a stable nonlinear dynamic input-output model of the business cycle. Such a model is still based on interindustry equilibrium and is of more interest than one which studies balanced growth.

CONCLUSION

In this study, we have outlined an input-output general equilibrium framework that enables us to control the relative strengths of backward and forward linkages of interindustry transactions in determining output and price levels. The conventional demand-side input-output model takes only the production (cost) structure of the economy. The iterative nature of the generalized input-output model, on the other hand, traces the feedbacks between demand and supply conditions and captures the effect of supply constraints on demand. When such supply constraints are due to upwardly rigid prices or price controls, our model has a kinship with the Keynesian theory of market disequilibrium as it would apply to interindustry transactions. We have, furthermore, shown that our analysis is compatable with production theory if there exists at least one value added category with no supply constraints.

Although we have formulated two mathematical structures for the generalized model, we have given a clear economic meaning to the horizontal model only, and our arguments here will apply to this version. We should also point out that a formal proof of the stability of the flexible price model remains to be shown.

An important feature of the model is its ability to describe a continuum of demand-side versus supply-side combinations for each sector. This leaves the problem of estimating the demand-supply mix parameters. Although our calculations in Chapter 2 can be extended to the more elaborate versions of Chapter 4, an econometric rather than programming approach is more appropriate. The magnitude and stability of these parameters need to be carefully considered when applying the model.

A fundamental point is that the same impacts give rise to different outcomes at different values of demand-supply mix parameters. Specifically, the effect of unbalanced growth on the performance of an economy will depend on the nature of the supply constraints. The generalized model captures these interactions and can assist policy makers in comparing alternative policy options and their impacts.

The model can be used to measure the relative historical importance of demand and supply factors in development. It remains to be investigated if any structural generalizations can be made for individual sectors over different stages of growth. An immediate application would then be in the identification of "key sectors" for development, an important technique often used by development planners. The objective of this approach is the measurement of sectoral linkage indices where the underlying principle is that the sectors with strong linkages should have investment priority in the development process, an idea first espoused by Hirschman (1958). Various linkage indices based on backward and forward multipliers (derived from the inverse matrices of demand- and supply-side models respectively) have been proposed, and a considerable amount of disagreement exists with regard to the proper way of combining the two classes of indices (see, for example, the debate in Quarterly Journal of Economics, Volume 90, 1976). This is due to treating the demand and supply determinants of linkages in separate frameworks. The generalized inverse matrix is an improved solution to the problem of the measurement of a total linkage index.

The input-output model constitutes the core of multisectoral computable general equilibrium (CGE) models. A promising extension of the generalized input-output is its incorporation into a CGE framework. This would allow for endogenous determination of final demand, connecting it to value added and prices, and permit us to extend the rationing scheme to components of value added and final demand categories. Furthermore, we can measure the welfare implications of loosening different supply constraints and set priorities in investment decisions.

Another extension is the incorporation of capital coefficients matrix to the von Neumann-Leontief model in Chapter 4 in order to investigate its "turnpike" properties. The demand-supply mix parameters embody the elements of market imperfections and to a degree are control variables. An interesting hypothesis, taking off from the Hirschmanian notion of forward linkages, is to test whether certain imperfections can lead to more desirable outcomes.

Finally, the construction of a generalized multiregional input-output model is of interest to regional scientists. Bon (1984) shows that the supply-side regional trade structure of the row coefficient multiregional model conceived by Polenske (1966) is incompatible with the demand-side regional technology structure and suggests (Bon (1988)) a model where both regional trade and technology is supply-driven. Using our framework, we can bridge Bon's model with the original Chenery-Moses column coefficient multiregional model.

In conclusion, the evaluation of the merits of our approach depends on further theoretical and empirical work.

APPENDIX 1: THE U.S. INPUT-OUTPUT DATA

The 23-sector U.S. input-output data that are used in the text are obtained from U.S. Bureau of the Census (1976) for the benchmark years 1947, 1958, 1963, and 1967 and from Miller and Blair (1985) for the benchmark years 1972 and 1977. The tables for the first four benchmark years are based on an industry-by-industry system of accounts where secondary production is treated as a sale by the producing sector to the sector to which the commodity belongs in its classification. Beginning in 1972, the Bureau of Economic Analysis adopted the commodity-by-industry system of accounts where two flow tables are presented: the Use matrix (U), a commodity-by-industry matrix where each column gives the amount of commodities used by a given industry, and the Make matrix (M), an industry-by-commodity matrix where each row gives the amount of commodities produced by a given industry. In order to make the 1972 and 1977 data compatible with the earlier industry-by-industry accounts, we have reworked it at the 23-sector level, prior to aggregation.[1] In this appendix, we describe how the latter system of accounts fits our formulation in the text.

The identities of this system are given by:

$$Q = Ui + Y = M^{T}i,$$

$$X = U^{T}i + W = Mi,$$

where

[1]For the years 1972 and 1977, data are aggregated to the 7-sector level after the transformations discussed below. First aggregating the Use and Make matrices, and then applying those transformations generates different results and introduces aggregation bias.

Q = n × 1 column vector of commodity outputs,

X = n × 1 column vector of industry outputs,

Y = n × 1 column vector of final demands for commodities,

W = n × 1 column vector of value added for industries.

The coefficient matrices are given by:

$$A = U\hat{X}^{-1},$$

$$B = \hat{Q}^{-1}U,$$

$$D = M\hat{Q}^{-1},$$

$$N = \hat{X}^{-1}M,$$

where

A is the commodity-by-industry direct-input coefficient matrix,

B is the commodity-by-industry direct-output coefficient matrix,

D is the market-shares matrix, where entries in each column show the proportion of a given commodity produced in each industry. (This assumption implies that commodities come in their own fixed proportions from various industries.)

N is the product-mix matrix, where entries in each row show the proportion a given industry produces of each commodity. (This assumption implies that each industry produces commodities in its own fixed proportions.)

Two possible assumptions can be made about secondary production using these coefficient matrices: that the secondary production behaves like its industry (industry-technology and industry-allocation) or that it behaves like the commodity group it belongs to (commodity-technology and commodity-allocation). The second option, which has been adopted by the United Nations System of National Accounts, can produce negatives in the inverse and is "unstable" in our framework. The first option, which we have adopted here, requires the assumption of fixed market-shares for the demand-side model, and the assumption of fixed product-mix for the supply-side model.

For the demand-side model:

$$Q = Ui + Y,$$

$$Q = \mathcal{A}X + Y,$$

$$Q = \mathcal{A}\mathcal{D}Q + Y,^{1}$$

$$Q = (I - \mathcal{A}\mathcal{D})^{-1}Y,$$

$$X = (I - \mathcal{D}\mathcal{A})^{-1}\mathcal{D}Y.$$

The following transformations, then, result in the industry-by-industry system of accounts we have used in the text:

$$A = \mathcal{D}\mathcal{A},$$

$$Z = \mathcal{D}U,$$

$$Y = \mathcal{D}Y.$$

For the supply-side model:

$$X = U^{T}i + \mathcal{W},$$

$$X = \mathcal{B}^{T}Q + \mathcal{W},$$

$$X = \mathcal{B}^{T}\mathcal{N}^{T}X + \mathcal{W},^{2}$$

$$X = (I - (\mathcal{N}\mathcal{B})^{T})^{-1}\mathcal{W},$$

and the similar transformations are given by:

---

[1] For the market-shares assumption: $X = Mi = \mathcal{D}\hat{Q}i = \mathcal{D}Q$.

[2] For the product-mix assumption: $Q = M^{T}i = \mathcal{N}^{T}\hat{X}i = \mathcal{N}^{T}X$.

$$B = \boldsymbol{N}\boldsymbol{B},$$

$$Z = M\boldsymbol{B},$$ [1]

$$W = \boldsymbol{W}.$$

Although these transformations result in a balanced industry-by-industry accounting framework described by identities (1) and (2), the treatment of secondary production (which makes up about 2.5 percent of total output at the 23-sector level) is much more accurate in the newer methodology and somehow undermines the conformity of the 1972 and 1977 tables to the earlier ones.

The 7- and 23-sector classifications are given in Table A.13.[2] The aggregated 7-sector U.S. input-output tables are shown in Table A.14. The values are millions of current dollars at producer's prices.

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[1] Note that $\boldsymbol{D}U = M\boldsymbol{B}$, and the demand- and supply-side formulations are consistent with each other because they generate the same industry-by-industry flow and coefficient matrices.

[2] For the related Standard Industrial Classification (SIC) codes, see Miller and Blair (1985, p.406).

Table A.13. Sectoral Classifications

7-sector Aggregation	23-sector Aggregation
1. Agriculture	Agriculture, Forestry, and Fishing
2. Mining	Metal Mining Petroleum and Natural Gas Mining Other Mining
3. Construction	Construction
4. Manufacturing	Food, Feed, and Tobacco Products Textile Products and Apparel Wood Products and Furniture Paper, Printing, and Publishing Chemicals and Chemical Products Petroleum and Coal Products Rubber, Plastics, and Leather Stone, Clay, and Glass Products Primary and Fabricated Metals Machinery, except Electrical Electrical Equipment and Supplies Transport Equipment and Ordnance Other Manufacturing
5. Transportation and Trade	Transportation and Trade
6. Services	Electric, Gas, and Sanitary Services Other Services
7. Other	Government Enterprises Scrap and Secondhand Goods

Table A.14. 7-sector U.S. Input-Output Data

1947

Interindustry Flow							Final Demand	Output
14741	0	92	23946	16	2048	15	6000	46858
47	810	277	6602	513	732	25	1376	10382
568	15	7	509	1489	4197	503	22043	29331
4435	952	11132	70687	4743	8643	668	96744	198004
2617	357	3884	10734	3617	4461	378	50596	76644
2981	997	1860	8300	9951	11083	193	45889	81254
5	7	32	2065	803	1938	6	-765	4091

Value Added

21464	7244	12047	75161	55512	48152	2303

1958

Interindustry Flow							Final Demand	Output
14806	0	237	25078	190	2304	624	8721	51960
102	1097	756	13349	37	1913	148	918	18320
613	11	8	752	2024	7842	1206	56835	69291
6109	1461	26543	131228	7977	23258	1575	154016	352167
2842	889	8446	21906	5041	8117	989	81102	129332
4338	2593	4261	21040	20817	30327	874	114575	198825
10	133	100	1974	2237	4940	12	877	10283

Value Added

23140	12136	28940	136840	91009	120124	4855

1963

Interindustry Flow							Final Demand	Output
17034	0	326	26752	260	2771	639	8908	56690
128	1111	737	14637	46	2727	189	967	20542
567	415	25	1400	1556	9556	1349	70445	85313
7649	1675	31588	179025	10172	22015	1687	205561	459372
2795	876	9789	24220	7244	10052	1553	103265	159794
4762	3501	5725	28828	23669	44491	1581	154788	267345
15	33	102	2555	2726	7371	14	1802	14618

Value Added

23740	12931	37021	181955	114121	168362	7606

1967

Interindustry Flow							Final Demand	Output
18542	0	263	31390	196	3014	392	9300	63097
138	1256	930	17280	42	3686	145	1454	24931
603	572	30	2560	1833	10328	1771	85583	103280
8679	2343	37563	232248	13705	33431	1938	278797	608704
4144	738	10839	30981	11447	15050	1498	141468	216165
5539	4258	7898	44695	33428	62483	2671	211937	372909
9	135	80	3371	3947	9310	23	2453	19328

Value Added

25443	15629	45677	246179	151567	235607	10890

1972

Interindustry Flow							Final Demand	Output
26429	1	313	40813	151	3095	19	13135	83956
186	1651	1524	22356	172	6170	326	-2000	30385
583	858	47	3244	3125	16464	2672	139005	165998
11961	2963	58610	288377	12227	49543	1217	336357	761225
4323	713	16834	48332	16196	13429	1126	190209	291163
8140	5173	12242	58992	46168	113170	3668	365565	613119
169	145	321	3500	2146	6610	302	143339	156535

Value Added

32165	18880	76107	295639	210976	404638	147206

1977

Interindustry Flow							Final Demand	Output
32059	9	696	62985	660	4977	172	28105	129662
370	5583	2387	77706	722	20239	1287	-30264	78030
1383	2923	304	8706	7182	32058	4971	206807	264335
25967	7478	98615	522462	30463	92596	3231	574171	1354983
7464	1736	28954	88542	30777	27418	2303	323445	510640
13046	9776	21105	100070	90714	187442	5894	651784	1079833
372	332	661	7325	3836	10696	795	222495	246512

Value Added

49002	50194	111616	487199	346287	704405	227859

APPENDIX 2: AN ALTERNATIVE MODEL THAT DOES NOT WORK

In this appendix, we consider an alternative closure to Equations (11) and (29). Oosterhaven (1981, p.146), while discussing the issue of forward and backward linkages, suggests the construct $T^* = \Lambda A + (I-\Lambda)B^T$ for the direct coefficient matrix. In our framework, the model is given by:

$$(90) \qquad \{ X^* = (I - \Lambda A^* - (I-\Lambda)B^{*T})^{-1}S^* : \quad S^* = \bar{S};\ \Lambda = \bar{\Lambda};$$
$$\Lambda A^*\hat{X}^* + (I-\Lambda)B^{*T}\hat{X}^* = \Lambda A\hat{X}^* + (I-\Lambda)B^T\hat{X}^* \}.$$ [1]

The model generates nonnegative values of X^* since $(I - T)$ is a nonsingular M-matrix.[2] However, difficulties emerge in constructing the interindustry flow matrix (and consequently the direct-input and direct-output coefficient matrices):

$$TX^* = (\Lambda A^* + (I-\Lambda)B^{*T})X^*,$$
$$= \Lambda A^*X^* + (I-\Lambda)B^{*T}X^*,$$
$$= \Lambda Z^* + (I-\Lambda)Z^{*T}.$$

Rearranging in order to solve for Z^*:

$$(91) \qquad (I- \Lambda)^{-1}\Lambda Z^* - Z^*(I-\Lambda)\Lambda^{-1} = (I-\Lambda)^{-1}(TX^*) - (X^*T)\Lambda^{-1}.$$

---

[1] This model is closer in structure to the conventional models as T is held constant over a projection ($T^* = T$), similar to the direct-input and direct-output coefficient matrices for demand- and supply-side models respectively.

[2] For this proof also, we use Theorem 2(III). Let $D = \hat{X}$, then

$$\begin{aligned}(I - T)\hat{X}i &= \hat{X}i - \Lambda A\hat{X}i - (I-\Lambda)B^T\hat{X}i,\\ &= X - \Lambda Zi - (I-\Lambda)Z^Ti,\\ &= X - \Lambda(X - Y) - (I-\Lambda)(X - W),\\ &= \Lambda Y + (I-\Lambda)W,\\ &> 0.\end{aligned}$$

This is an equation in the form of $\Phi Z^* + Z^* \Theta = \Omega$, where:

$\Phi = (I-\Lambda)^{-1}\Lambda$,

$\Theta = -(I-\Lambda)\Lambda^{-1}$,

$\Omega = (I-\Lambda)^{-1}(TX^*) - (X^*T)\Lambda^{-1}$.

There are several solutions suggested in the literature for this particular matrix linear equation.[1] The most convenient method, due to the diagonal nature of Φ and Θ, is defining the column string of Z^* as:

$$csZ^* = [z_{11}\ldots z_{n1}z_{12}\ldots z_{n2}\ldots z_{1n}\ldots z_{nn}]^T,$$

and the column string of Ω, $cs\Omega$, analogously. The solution is given by:

$$csZ^* = \Psi^{-1}cs\Omega, \tag{92}$$

where

$\Psi = I \otimes \Phi + \Theta^T \otimes I$,

$\otimes$ stands for the tensor (Kronecker) product.

Existence of the solution to Equation (92) hinges upon the nonsingularity of the diagonal matrix Ψ whose n^2 eigenvalues are given by $((1-\lambda_i)^{-1}\lambda_i - (1-\lambda_j)\lambda_j^{-1})$ for all combinations of i and j. Hence, the solution exists only if none of the eigenvalues are zero:

$$\lambda_i/(1-\lambda_i) \neq (1-\lambda_j)/\lambda_j$$

$$\lambda_i\lambda_j \neq (1-\lambda_j)(1-\lambda_i)$$

$$\lambda_i + \lambda_j \neq 1, \qquad i,j = 1,\ldots,n.$$

This implies that, when any two sectoral demand-supply mix parameters add to 1, the model "blows up." In fact, whenever their sum approaches 1 from either direction, the model tends to produce nonsensical interindustry

[1] For a comprehensive theory, see Kucera (1974).

flow and direct coefficient matrices.[1] The problem arises when the model stretches A^* and B^* in unrealistic (and opposite) directions in order to keep T constant. Hence, the structure given by (90) is unstable, and the model cannot generate a meaningful interindustry flow matrix because it fails to span a "continuous space" between the demand- and supply-side models for this matrix.

[1] Note that if a given sector has $\lambda_i = 0.5$, the fact that $\lambda_i + \lambda_i = 1$ makes the matrix Ψ singular.

APPENDIX 3: SOLUTIONS TO THE VON NEUMANN-LEONTIEF MODEL

Assume a hypothetical economy given by the base year technology matrix A. Under the balanced von Neumann-Leontief model, we obtain the following growth solution:

$$A = \begin{bmatrix} 0.4 & 0.0 & 0.1 \\ 0.0 & 0.1 & 0.8 \\ 0.5 & 0.7 & 0.1 \end{bmatrix}$$

$$\gamma = \beta = 1.1111,$$

$$x^T = [0.0909 \;\; 0.4545 \;\; 0.4545],$$

$$p = [0.3478 \;\; 0.3043 \;\; 0.3478].$$

Table A.15 gives selected solutions of quadruplets $\{G, \beta, x, p\}$ for the demand-side, supply-side, horizontal and vertical unbalanced growth von Neumann-Leontief models respectively. For both generalized models, we specify $\lambda_1 = 0.25$, $\lambda_2 = 0.50$, and $\lambda_3 = 0.75$.

Table A.15. Solutions to the Unbalanced von Neumann-Leontief Growth Model

DEMAND-SIDE

Growth Factor Vector			Output Proportions			β	Price Proportions		
1.0000	1.0000	1.2414	0.0896	0.4776	0.4328	1.1111	0.3478	0.3043	0.3478
1.0000	1.0682	1.1723	0.0870	0.4676	0.4454	1.1111	0.3478	0.3043	0.3478
1.0000	1.1363	1.1095	0.0846	0.4580	0.4574	1.1111	0.3478	0.3043	0.3478
1.0000	1.2045	1.0524	0.0822	0.4488	0.4689	1.1111	0.3478	0.3043	0.3478
1.0000	1.2727	1.0000	0.0800	0.4400	0.4800	1.1111	0.3478	0.3043	0.3478
1.1810	1.0000	1.1988	0.0991	0.4648	0.4361	1.1111	0.3478	0.3043	0.3478
1.1810	1.0682	1.1342	0.0964	0.4554	0.4482	1.1111	0.3478	0.3043	0.3478
1.1810	1.1363	1.0754	0.0937	0.4464	0.4599	1.1111	0.3478	0.3043	0.3478
1.1810	1.2045	1.0216	0.0912	0.4377	0.4711	1.1111	0.3478	0.3043	0.3478
1.3620	1.0000	1.1470	0.1109	0.4488	0.4402	1.1111	0.3478	0.3043	0.3478
1.3620	1.0682	1.0877	0.1080	0.4402	0.4518	1.1111	0.3478	0.3043	0.3478
1.3620	1.1363	1.0335	0.1051	0.4319	0.4630	1.1111	0.3478	0.3043	0.3478
1.5431	1.0000	1.0825	0.1260	0.4286	0.4454	1.1111	0.3478	0.3043	0.3478
1.5431	1.0682	1.0296	0.1228	0.4209	0.4564	1.1111	0.3478	0.3043	0.3478
1.7241	1.0000	1.0000	0.1457	0.4020	0.4523	1.1111	0.3478	0.3043	0.3478

SUPPLY-SIDE

Growth Factor Vector			Output Proportions			β	Price Proportions		
1.0000	1.0000	1.2414	0.0811	0.4324	0.4865	1.1111	0.3731	0.3264	0.3005
1.0000	1.0682	1.1723	0.0785	0.4506	0.4709	1.1111	0.3742	0.3065	0.3192
1.0000	1.1363	1.1095	0.0760	0.4678	0.4561	1.1111	0.3744	0.2883	0.3374
1.0000	1.2045	1.0524	0.0737	0.4843	0.4420	1.1111	0.3736	0.2714	0.3550
1.0000	1.2727	1.0000	0.0714	0.5000	0.4286	1.1111	0.3721	0.2558	0.3721
1.1810	1.0000	1.1988	0.1060	0.4207	0.4733	1.1111	0.3313	0.3423	0.3264
1.1810	1.0682	1.1342	0.1027	0.4388	0.4586	1.1111	0.3324	0.3215	0.3461
1.1810	1.1363	1.0754	0.0995	0.4559	0.4445	1.1111	0.3325	0.3024	0.3651
1.1810	1.2045	1.0216	0.0965	0.4723	0.4311	1.1111	0.3318	0.2847	0.3836
1.3620	1.0000	1.1470	0.1368	0.4062	0.4570	1.1111	0.2959	0.3527	0.3514
1.3620	1.0682	1.0877	0.1326	0.4241	0.4433	1.1111	0.2969	0.3313	0.3718
1.3620	1.1363	1.0335	0.1287	0.4412	0.4301	1.1111	0.2970	0.3115	0.3914
1.5431	1.0000	1.0825	0.1759	0.3878	0.4363	1.1111	0.2649	0.3576	0.3775
1.5431	1.0682	1.0296	0.1708	0.4054	0.4238	1.1111	0.2658	0.3359	0.3983
1.7241	1.0000	1.0000	0.2273	0.3636	0.4091	1.1111	0.2363	0.3564	0.4073

HORIZONTAL

Growth Factor Vector			Output Proportions			β	Price Proportions		
1.0000	1.0000	1.3068	0.0814	0.4650	0.4536	1.1397	0.3641	0.3130	0.3229
1.0000	1.0529	1.2298	0.0802	0.4646	0.4552	1.1308	0.3613	0.3078	0.3309
1.0000	1.1059	1.1531	0.0791	0.4641	0.4568	1.1193	0.3574	0.3029	0.3397
1.0000	1.1588	1.0765	0.0779	0.4637	0.4585	1.1050	0.3524	0.2983	0.3493
1.0000	1.2118	1.0000	0.0767	0.4632	0.4601	1.0879	0.3464	0.2939	0.3597
1.1441	1.0000	1.2574	0.0974	0.4520	0.4506	1.1330	0.3523	0.3159	0.3318
1.1441	1.0529	1.1821	0.0962	0.4517	0.4521	1.1229	0.3495	0.3104	0.3401
1.1441	1.1059	1.1069	0.0949	0.4514	0.4536	1.1102	0.3457	0.3052	0.3491
1.1441	1.1588	1.0318	0.0937	0.4511	0.4552	1.0947	0.3408	0.3003	0.3588
1.2883	1.0000	1.1939	0.1167	0.4361	0.4472	1.1242	0.3420	0.3170	0.3410
1.2883	1.0529	1.1205	0.1154	0.4360	0.4487	1.1127	0.3391	0.3113	0.3496
1.2883	1.1059	1.0473	0.1140	0.4359	0.4501	1.0984	0.3352	0.3059	0.3589
1.4324	1.0000	1.1108	0.1404	0.4161	0.4435	1.1121	0.3322	0.3163	0.3514
1.4324	1.0529	1.0400	0.1390	0.4162	0.4448	1.0985	0.3290	0.3105	0.3605
1.5766	1.0000	1.0000	0.1703	0.3904	0.4392	1.0938	0.3220	0.3134	0.3646

VERTICAL

Growth Factor Vector			Output Proportions			β	Price Proportions		
1.0000	1.0000	1.1844	0.0862	0.4596	0.4542	1.0858	0.3613	0.3080	0.3307
1.0000	1.0833	1.1384	0.0839	0.4601	0.4560	1.1017	0.3653	0.2986	0.3361
1.0000	1.1666	1.0924	0.0816	0.4606	0.4578	1.1148	0.3676	0.2906	0.3418
1.0000	1.2500	1.0462	0.0793	0.4611	0.4597	1.1253	0.3684	0.2837	0.3480
1.0000	1.3333	1.0000	0.0769	0.4615	0.4615	1.1334	0.3677	0.2777	0.3546
1.2102	1.0000	1.1583	0.1022	0.4476	0.4502	1.0951	0.3283	0.3235	0.3482
1.2102	1.0833	1.1137	0.0997	0.4485	0.4519	1.1105	0.3324	0.3138	0.3539
1.2102	1.1666	1.0690	0.0971	0.4492	0.4536	1.1230	0.3348	0.3054	0.3599
1.2102	1.2500	1.0242	0.0945	0.4500	0.4555	1.1330	0.3357	0.2980	0.3663
1.4204	1.0000	1.1232	0.1233	0.4319	0.4448	1.1036	0.3024	0.3346	0.3630
1.4204	1.0833	1.0805	0.1205	0.4331	0.4464	1.1183	0.3063	0.3248	0.3690
1.4204	1.1666	1.0376	0.1176	0.4342	0.4482	1.1301	0.3087	0.3161	0.3752
1.6307	1.0000	1.0739	0.1525	0.4102	0.4372	1.1127	0.2810	0.3423	0.3767
1.6307	1.0833	1.0337	0.1493	0.4119	0.4388	1.1263	0.2847	0.3324	0.3829
1.8409	1.0000	1.0000	0.1955	0.3786	0.4259	1.1236	0.2624	0.3465	0.3911

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